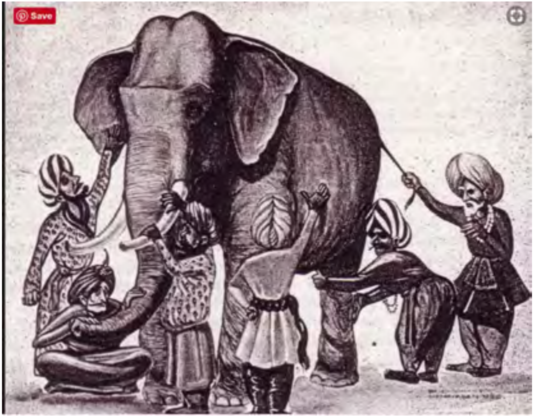
The work of Mike Hochman on multidimensional symbolic dynamics and Borel dynamics

Mike Boyle

Department of Mathematics
University of Maryland

The elephant



Hochman's Elephant

I. Fractals and rigidity
II. Multidimensional symbolic dynamics
III. Borel dynamics
IV. Ergodic theory
V. Topological dynamics
VI. Percolation theory

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I'll only give a quick look at the symbolic elephant, and the Borel elephant.

I could try more, but sometimes,

“The more I do, the less I do.”

Multidimensional symbolic dynamics

I'll single out the content and impact of two papers:

[HM] Michael Hochman and Tom Meyerovitch.
A characterization of the entropies of multidimensional shifts of finite type. Annals of Math, 2010. (arXiv 2007)

[H1] Michael Hochman.
On the dynamics and recursive properties of multidimensional symbolic systems. Inventiones Math, 2009. (preprint 2008)

Definitions

"Multidimensional symbolic dynamics" refers to Z^d subshifts, (X, σ) , $d = 2, 3, ...$.

An element x of X is a function from Z^d into some finite alphabet A; x is visualized as a way of filling the Z^d lattice with symbols from A.
Let $C_n(x)$ be the restriction of x to the cube $\{0, 1, ..., n-1\}^d$.
The Z^d topological entropy of (X, σ) is

$$\lim(1/(n^d)\log|\{C_n(x) : x \in X\}|.$$

A d dimensional cellular automaton is a continuous shift-commuting map from a full Z^d shift into itself.

Before **[HM]** :

• ($d = 1$) The possible topological entropies of a Z SFT are logs of an easily understood class of algebraic integers.

• A limited collection of Z^d entropies computed, by mathematical physicists (Baxter, Lieb ...).

• Known: for $d > 1$, there is no algorithm to compute entropy of Z^d SFTs. (Or, roughly, anything nontrivial about them.)

Theorem [HM] Suppose h is a nonnegative real number and d is an integer, $d \geq 2$.
TFAE.
(1) h is the topological entropy of a Z^d shift of finite type.
(2) There is a Turing machine which produces a sequence of nonnegative numbers h_n such that $h = \inf h_n$.

The countable class of numbers in (2) properly contains the nonnegative real numbers h which can be algorithmically approximated to arbitrary precision.

E.g. algebraic numbers, π, e ...

The paper [H1] showed this theorem was part of a systematic approach.

Theorem [H1] Suppose h is a nonnegative real number and d is an integer, $d \geq 3$. TFAE.
(1) h is the topological entropy of a Z^d cellular automaton.
(2) There is a Turing machine which produces a sequence of nonnegative numbers h_n such that $h = \liminf h_n$.

• For $h = \inf h_n$, at least h_n gives you an upper bound to h; for $h = \liminf h_n$, you have no bound at all from h_n.

• There are deeper results in [H1], e.g. giving (for $d \geq 3$) a recursive theoretic characterizations of the possible dynamics of a d-dimensional cellular automaton on its limit set (up to a modest equivalence relation).

Impact of [HM] and [H1]

Before [H1,HM], workers were very aware of the "swamp of undecidability" [Lind] as an obstacle to finding theorems about c.a. or Z^d SFTs.

[H1,HM] did "mathematical judo" on recursion theory: making it a friend instead of an enemy.

[H1,HM] gave a blueprint for characterizing possible properties of problems for effective Z^d symbolic dynamics:
– find the "obvious" recursion theoretic obstruction
– make constructions to prove there are no others.

Fine work by various people has since been carried out in this vein.

It now seems that to a large extent the landscape of possibilities for general multidimensional SFTs (or sofic shifts, or effective shifts) has a recursion theoretic description.

It seems fair to say that the papers [HM,H1] changed the way people thought about the subject.

Also:
the papers [HM], [H1] influenced not only how people thought about the subject, but also who thought about it. Some workers based in logic/recursion theory, or computer science, came into the subject, which is now active and interdisciplinary.

Lastly:
those papers gave powerful construction techniques, which had been lacking.
One tool for these was the Shahar Mozes (1989) construction (using ideas from Raphael Robinson (1971)) of $\mathbb{Z}^2$ SFT presentations for many planar substitution tilings.

Borel dynamics

Dynamics is rich with multiple viewpoints.

E.g. we might consider a homeomorphism $T : X \to X$ with respect to a differentiable structure; with respect to an invariant probabiity of interest; or as a Borel system.

DEFN A standard Borel space is a pair (X, Σ) such that X is a set, and Σ is a σ-algebra (its Borel σ-algebra) generated by open sets of some complete, separable metric on X.

A morphism in the Borel category is a map for which the inverse image of every Borel set is a Borel set.
Standard Borel spaces of equal cardinality are isomorphic.

DEFN A Borel system (X, T) is an automorphism $T : X \to X$ of a standard Borel space.

Borel dynamics off null sets

[H2] Michael Hochman. Isomorphism and embedding of Borel systems on full sets. Acta Appl. Math. (Kim memorial volume) 2013. (arXiv 2010).

We often try to understand a system (X, T) in terms of some invariant probability, i.e., neglecting its measure zero sets.

More ambitiously: [H2] considers understanding a system in terms of ALL its invariant probabilities: i.e., neglecting only its **null sets**: sets of measure zero for **all** invariant Borel probabiities.
(A full set is the complement of a null set.)

For familiar systems, already the periodic point data (cardinality of size n orbits), $n \in N$,
is quite varied, even for systems of equal entropy. So one might guess that on the complement of the periodic points (MUCH more complicated) the possibilities for the Borel dynamics modulo null sets would be hopelessly varied. Hochman showed this is not the case within many familiar classes.

For $t > 0$, let B_t be the collection of Borel systems which are free (no periodic points) and with no invariant Borel probability of entropy $\geq t$.

DEFN [H2] A Borel system (X, T) is strictly t-universal if it is in B_t and every member of B_t embeds modulo null sets to a subsystem of (X, T).

DEFN [H2] The entropy of a Borel system (X, T) is the supremum of $h_\mu(T)$, over T-invariant Borel probabilities μ.

For a Borel system (X, T), form X' by removing periodic points and points generic for any ergodic measure of maximal entropy.

Theorem [H2] Suppose a Borel system (X, T) of finite entropy $t > 0$ contains mixing SFTs with entropies arbitrarily close to t.
(1) Then (X', T) is strictly t-universal.
(2) Strictly t-universal systems are isomorphic mod null sets.

Proof.
(2) Not hard. (Cantor-Bernstein argument of set theory.)
(1) As follows:

(I) Broad strategy B. Weiss showed a free Borel system has a countable generator, hence embeds to a subsystem of $((1,2,...)^Z,\sigma)$, the full shift over a countable alphabet.
Then, it's not hard to show:
given $t > t-\epsilon > 0$ and a mixing SFT of entropy t:
it suffices to find B a Borel subsystem of $((1,2,...)^Z,\sigma)$ supporting all the ergodic measures of entropy $\leq t-\epsilon$,
and Borel embed B into that SFT.
(II) Finer strategy. "Observe": the argument of the Krieger generator theorem gives a Borel map which on the generic points of every ergodic Borel measure, individually, is injective with finitary inverse.
(III) Hard construction step. Augment the coding of that map to make it injective on the union of all those supports.

So!

Corollary [H2] For every (X, T) of equal positive entropy t from the following collection, (X', T) is the same strictly t-universal system:
mixing SFTs, mixing countable state Markov shifts, mixing finitely presented systems, Anosov diffeomorphisms ...

Each of those systems above has a unique measure of maximal entropy, which is Bernoulli; then X is the disjoint union of X', the periodic points, and the set of points generic for the maximal measure.

Corollary [H2] Any two systems of equal positive entropy from the list above are Borel isomorphic modulo null sets and periodic points.

Before [H2]:

Motivated from low dimensional (piecewise) smooth dynamics:

Defn (Buzzi, 1997) Suppose S,T have equal and finite topological entropy h. S and T are entropy conjugate if for some $\epsilon > 0$ they are isomorphic modulo sets of measure zero for all ergodic invariant Borel probabilities of entropy $\geq h - \epsilon$.

(Defn related to, but not the same as, Bowen's "entropy conjugacy" (1973).) Buzzi asked if equal entropy mixing SFTs, or more generally positive recurrent countable state Markov shifts, must be entropy conjugate.

Positive answers for SFTs (via 1979 Adler-Marcus Theorem) and strong positive recurrent countable state Markov shifts (B-Buzzi-Gomez, Crelles' J. 2006) used explicit symbolic construction.

Again, we see work of Hochman which not only answers a resistant problem, but gives a more insightful viewpoint, leading to more.

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I cannot help but wonder if t-universal sets arise in even broader classes. Does a quasihyperbolic toral automorphism of $\mathbb{T}^4$ admit embeddings mod null sets of all free lower entropy Borel systems?

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Away from mixing systems, the universality approach can become more complicated, but still useful. E.g,
in [B-Buzzi-Gomez, JEMS 2017]:
Every $C^{1+\alpha}$ surface diffeomorphism is Borel isomorphic to a countable state Markov shift modulo sets of measure zero for all ergodic positive entropy measures.

Borel dynamics, beyond probabiities

[H3] Michael Hochman.
Every Borel automorphism without finite invariant measure admits a two-set generator.
JEMS, to appear (arXiv 2015)

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For a Borel system (X, T), pick a countable collection of sets generating the Borel σ-algebra and let X_p be the set of points which visit these sets with ergodic-theorem frequencies for some T-invariant ergodic probability on X.

[H2] considered T on X_p. What about the complement?

Given a Borel system (X, T), let $X_d = X \setminus X_p$.

The system (X_d, T) supports no invariant probability. But when (X_d, T) is not a wandering system, it supports other ergodic theory (e.g. infinite invariant measures) and complicated dynamics.

I think of X_d as the “dark matter”.
(Usually there is a lot of it, and the dynamics of (X_d, T) is harder to see.)

Let D be the class of Borel systems admitting no invariant Borel probability.

The study of the class D began with Shelah and Weiss (SW 1982, W84, W1989), and then Kechris and others.
Earlier, Krengel had shown that for any infinite measure preserving ergodic conservative system, there is a two-set generator. (This is a.e., neglecting measure zero sets.)

Question (Weiss, 1989) Suppose a Borel system admits no finite invariant Borel probability. Must it have a finite generator? A 2-set generator?

This was a rather fundamental, longstanding question. (Compare, Krieger's finite generator theorem [1970].)

The title of **[H3]** gives the answer.

Note and compare:

Theorem (Tserunyan) 2015 (2012 arxiv)
Suppose G is an arbitary countable group, acting by homeomorphisms on a σ-compact Polish space. Then G has a finite generator. (In fact, a 32-generator.)

This theorem is remarkable for the generality of the acting group.

However, not every $\mathbb{Z}$-action on a Borel space is Borel conjugate to a continuous action on a σ-compact space.
The two theorems seem quite different.
The proofs are very different.

Proofs: [H3] is much harder than [H2].
One big problem is to even find a strategy. See [H3].
The **[H3]** proof leads to compelling open problems (see **[H3]**).

Theorem [H3] If a Borel system (X, T) contains a nontrivial mixing SFT, then (X_d, T) is the unique (up to Borel conjugacy) Borel system in D into which every system in D embeds.

So, for many familiar systems, (X_d, T) is the same universal "dark matter" Borel dynamics. (Whatever that is!)

It is natural to wonder how widely this universal system arises. (In a quasihyperbolic toral automorphism of $\mathbb{T}^4$?)

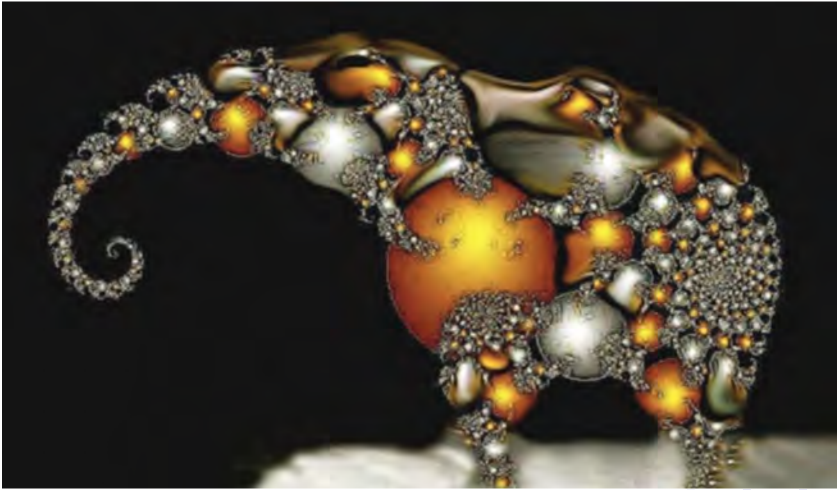
Corollary [H3] Suppose two homeomorphisms of equal finite positive entropy are mixing and lie in any of following classes: SFT, sofic shift, positive recurrent countable state Markov shift, finitely presented system (e.g. Anosov homeomorphism, or Axiom A on a basic set).

Then their restrictions to the complement of the periodic points are Borel isomorphic.

That ends our quick tour of the symbolic dynamics elephant, and the Borel dynamics elephant.

But you will see more ...

The fractal elephant



Fractal Elephant (by Julie Grace)