Sébastien Gouëzel, problems solver

François Ledrappier
LPSM, Sorbonne Université

Brin Prize 2019, September 28, 2019
1. Local limit theorems for symmetric random walks in Gromov hyperbolic groups, *J.A.M.S.* 27 (2014) 893–928,

2. Subadditive and multiplicative ergodic theorems (with A. Karlsson), *preprint*.

1. LLT in hyperbolic groups

**Theorem 1** Let $G$ be a non-elementary Gromov hyperbolic group, $\mu$ a symmetric probability measure on $G$.
We assume that the support of $\mu$ is finite, that it generates $G$ as a group and is aperiodic. Consider the $n$-fold convolution $\mu^{(*n)}$. Then there are a number $R > 1$ and a positive function $C$ on $G$ such that, as $n \rightarrow \infty$,

$$\mu^{(*n)}(x) \sim C(x)R^{-n}n^{-3/2}.$$
Remarks:

• By Kesten 1959, $-\frac{1}{n}\log \mu^{(*n)}(x) \rightarrow \log R > 0$, since $G$ is not amenable.

• Known for the free groups (S. Lalley, see below). ‘3/2’ also fits with the results of P. Bougerol (1981) about Brownian motion on symmetric spaces.

• The function $C$ satisfies

\[
R^{-1}C(x) = \sum_{y\in G} C(y^{-1}x)\mu(y),
\]

i.e. $C$ is a positive $R^{-1}$-eigenfunction for the Markov operator $PF(x) := \sum_{y\in G} F(y^{-1}x)\mu(y)$.

The proof uses the resolvant $(I-d - rP)^{-1}$ as $r \rightarrow R_-$.
Set $G_r(x) := \sum_{n \geq 0} r^n \mu^{(*n)}(x) < +\infty$ for $r < R$.

**Fact** $G_R(x) := \sum_{n \geq 0} R^n \mu^{(*n)}(x) < +\infty$ as well.

(An observation by Y. Guivarc’h and by D. Sullivan)

**Theorem 2** Same setting as Theorem 1. Then, there exists a positive function $C'$ on $G$, such that, as $r \to R_-$,

$$\frac{dG_r(x)}{dr} \sim - \frac{C'(x)}{\sqrt{R - r}}.$$ 

Theorem 2 $+$ Karamata Theorem $+$ some regularity of $\mu^{(*n)}(x)$ in $n \Rightarrow$ Theorem 1.
Remarks:

• The symmetry of $\mu$ is used in the Tauberian argument. S.G. conjectures that it is not necessary. See the Appendix of the paper for results in the case of surface groups without assuming $\mu$ symmetric.

• It is relatively easy to see that $\frac{dG_r(x)}{dr}$ is comparable to $(R - r)^{-1/2}$. The point of Theorem 2 is that there is a precise equivalent. S. Lalley proved Theorem 2 for the free groups by observing that the $G_r(x)$ satisfy algebraic equations and then applying Puiseux Theorem. There is no similar argument for other hyperbolic groups.
A geometric proof in the free group case would involve the following ingredients

a) Assume \( y \) is a point on the segment \([e, x]\). Since its steps are bounded, a trajectory from \( e \) to \( x \) has to pass near \( y \). In particular, there is a constant \( D \) such that, uniformly in \( r, R/2 \leq r \leq R \),

\[
G_r(x) \leq DG_r(y)G_r(y^{-1}x).
\]

b) The coding of the boundary by a subshift of finite type. Then, for a continuous family of Hölder potentials, the equilibrium measures depend continuously and are uniformly 2- and 3-mixing.

c) Use geometric estimates, a) and b) to see that a variant \( \eta(r) \) of \( \frac{dG_r(e)}{dr} \) satisfies \( \eta'(r) \sim C''(\eta(r))^3 \) as \( r \to R_- \).
For Fuchsian groups, there is a Markov coding of the boundary by Bowen-Series, and the analog of b) still holds.

What is known about a) comes from the work of Ancona, who showed that there is \( \alpha(r) > 0, D(r) \) such that

\[
G_r(x) \leq D(r)e^{-\alpha(r)|x|}
\]

and, if \( y \) is a point on the segment \([e,x]\),

\[
G_r(x) \leq D(r)G_r(y)G_r(y^{-1}x).
\]
When $r \to R_-$, Ancona’s $(\alpha(r))^{-1}$ and $D(r)$ are unbounded. S.G. and S. Lalley give a proof of *uniform* Ancona inequalities, so that the geometric arguments from c) can be extended. They prove a uniform superexponential estimate for the contribution to $G_R(x)$ of the trajectories that avoid a ball of radius $|x|/100$ around the middle of $[e,x]$. Uniform Ancona inequalities follow by a nested induction argument.
For the general hyperbolic group, this latter argument is extended in [G14] by a very nice combinatorial proof.

But the symbolic codings given by Coornaert-Papadopoulos or Calegari-Fujiwara are not transitive, let alone mixing. [G14] also shows that, for the potentials considered, all transitive components of the CF coding have the same thermodynamical formalism.
Remarks and extensions

1. It is also proven in [G14] that the Martin boundary at $R$ is the geometric boundary, i.e. $\frac{G_R(z^{-1}y)}{G_R(z^{-1})}$ converges iff $z \to \xi \in \partial G$.

2. This is still true if $\mu$ satisfies $\sum_{x \in G} e^{\delta|x|} \mu(x) < +\infty$ for all $\delta$, but not when $\sum_{x \in G} e^{\delta|x|} \mu(x) < +\infty$ only for some $\delta$ ([G15]).

3. Analog results hold for the heat kernel on the universal cover of a compact negatively curved manifold (L.-Lim, preprint).
2. Subadditive and multiplicative ergodic theorems (with A. Karlsson)

**Theorem** (Oseledets) Let $(\Omega, A, m, T)$ be an ergodic measurable dynamical system, $A$ a measurable mapping $A : \Omega \to GL(d, \mathbb{R})$ such that $\int \log^+ A(\omega) \, dm(\omega) < +\infty$. Form $A^{(n)}(\omega) := A(\omega) \cdots A(T^{n-1}\omega)$. Then, $\left( A^{(n)}(\omega)(A^{(n)}(\omega))^\ast \right)^{1/2n}$ converge.

We recall Kaimanovich’s reading of the Raghunathan-Ruelle proof. Firstly, we have Kingman subadditive ergodic theorem:
**Theorem** (Kingman)
Let \( a(n, \omega) \) be a sequence of numbers satisfying
\[
a(n + m, \omega) \leq a(n, \omega) + a(T^n \omega, m) \quad \text{and} \quad \int a^+(1, .) \, dm < +\infty,
\]
Then, for \( m \)-a.e. \( \omega \), as \( n \to \infty \),
\[
\frac{1}{n} a(n, \omega) \to \inf \frac{1}{n} \int a(n, \omega) \, dm.
\]

Let \( GL(d, \mathbb{R}) \) act on the space \( X \) of symmetric matrices by \( A(S) = A^* SA \) and let \( \alpha := \lim_{n \to \infty} \frac{1}{n} d(e, A^{(n)}(\omega)e) \).

**Proposition**  Assume \( \alpha > 0 \). Then, for a.e. \( \omega \), any \( y \in X \), there is a unique geodesic \( \gamma \) in \( X \) such that
\[
\gamma(y) = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} d(A^{(n)}(\omega)y, \gamma(n\alpha)) = 0. \quad (*)
\]

V. Kaimanovich: Kingman \( \Rightarrow \) Proposition and (*) is a reformulation of \((A^{(n)}(\omega)(A^{(n)}(\omega))^*)^{1/2n}\) converge.
We are interested in composition of non-linear transformations.

**Theorem** [Karlsson-Margulis 99] Assume the uniformly convex metric space $X$ is non-positively curved in the sense of Busemann and let $A(\omega)$ be a measurable family of semicontractions of $X$ such that

\[
\int d(A(\omega)x, x) \, dm < \infty \text{ for some (all) } x \in X.
\]

Form $A^{(n)}(\omega) := A(\omega) \cdots A(T^{n-1}\omega)$ and assume that

\[
\alpha := \lim_{n \to \infty} \frac{1}{n} d(x, A^{(n)}(\omega)x) > 0 \text{ for some (all) } x.
\]

Then, for a.e. $\omega$, any $x \in X$, there is a unique geodesic $\gamma$ in $X$ such that

\[
\gamma(x) = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} d(A^{(n)}(\omega)x, \gamma(n\alpha)) = 0.
\]
Uniformly convex means that the middle \( m_{x,y} \) exists and is definitely closer to any other point \( w \) than \( x \) or \( y \) and non-positively curved that the middles \( m_{xy}, m_{xz} \) satisfy \( d(m_{xy}, m_{xz}) \leq 1/2d(x,y) \). CAT(0) spaces, \( L^p \) spaces, for \( 1 < p < +\infty \), are uniformly convex and non-positively curved in the sense of Busemann. Semi-contraction means \( d(Ax, Ay) \leq d(x,y) \).

An application of [KM]Theorem is the extension by Grabarnik and Guysinsky of Kalinin’s Lifschitz theorem to invertible elements in a Banach ring [D.C.D.S. 17].
The proof of [KM]Theorem rests on a refinement of Kingman subadditive ergodic theorem.

**Theorem** [Karlsson-Margulis 99] Let $a(n, \omega)$ be a subadditive cocycle and assume that

$$\alpha := \inf \frac{1}{n} \int_{n}^{\infty} a(n, \omega) \, dm > -\infty.$$ 

Then, for a.e. $\omega$, every $\epsilon > 0$, there exist $K(\omega, \epsilon)$ and infinitely many $n$ such that for all $k, K \leq k \leq n$, 

$$a(n, \omega) - a(n - k, T^k \omega) \geq (\alpha - \epsilon)k.$$
[GK2?] consider a general metric space. Fix $x_0 \in X$ and for $x \in X$, set

$$h_x(y) := d(y, x) - d(x_0, x).$$

**Definition** A metric functional is a pointwise limit of functions $h_x$.

Called *horofunctions* if $X$ is proper and the convergence is uniform on compact sets, *Busemann functions* if $X$ is a geodesic space and $x$ goes to infinity along a geodesic.
**Theorem 3** [Gouëzel-Karlsson] Let $X$ be a metric space and $A(\omega)$ be a measurable family of semicontractions of $X$ such that

$$\int d(A(\omega)x, x) \, dm < \infty$$

for all $x \in X$.

Form $A^{(n)}(\omega) := A(\omega) \cdots A(T_{n-1}^\omega)$ and assume that

$$\alpha := \lim_{n \to \infty} \frac{1}{n} d(x, A^{(n)}(\omega)x) > 0$$

for all $x$.

Then, for a.e. $\omega$, there exists a metric functional $h_\omega$ such that

$$\alpha = \lim_{n \to \infty} -\frac{1}{n} h_\omega(A^{(n)}(\omega)x)$$

for all $x$. 

18
The proof of Theorem 3 rests on a further refinement of Kingman theorem.

**Theorem 4** [Gouëzel-Karlsson] Let $a(n, \omega)$ be a subadditive cocycle and assume that

$$\alpha := \inf_n \frac{1}{n} \int a(n, \omega) \, dm > -\infty.$$

Then, for a.e. $\omega$, there exist integers $n_i = n_i(\omega)$ and positive real numbers $\delta_\ell = \delta_\ell(\omega), \delta_\ell(\omega) \to 0$ as $\ell \to \infty$ such that for every $i$ and every $\ell \leq n_i$,

$$\left| a(n_i, \omega) - a(n_i - \ell, T^\ell \omega) - \alpha \ell \right| \leq \ell \delta_\ell(\omega).$$

Moreover, on a set of large measure, one can take $\delta$ uniform and have a large density of good times $n_i$. 
Kalinin and Sadovskaya (E.T.D.S. 19) use Theorem 4 to extend to cocycles over a hyperbolic system with values in operators in a Banach space the finite dimensional result of Kalinin’s that the maximal and minimal Lyapunov exponents are approximated by the norms of the cocycle and of its inverse at periodic points.
3. Small eigenvalues of the Laplacian for algebraic measures in moduli space, and mixing properties of the Teichmüller flow (with A. Avila)

Let $S$ be a surface of genus $g$, $\Sigma = \{\sigma_1, \cdots, \sigma_j\}$ a set of punctures with multiplicities $K := \{\kappa_1, \cdots, \kappa_j\}$ such that $\sum (\kappa_i - 1) = 2g - 2$, $\text{Teich}(S, \Sigma, K)$ (resp. $\text{Teich}_1(S, \Sigma, K)$) the corresponding space of abelian differentials (resp. of area 1), $\Gamma$ the mapping class group, $X$ the quotient space $\text{Teich}_1(S, \Sigma, K)/\Gamma$. 
$SL(2, \mathbb{R})$ acts on $X$, defines a foliation of $X$ into orbits. On the quotient by $SO(2, \mathbb{R})$, this action defines a foliation of $SO(2, \mathbb{R}) \setminus X$ into (quotients of) hyperbolic planes. Let $\mu$ be a probability measure on $X$ that is invariant ergodic under the action of $SL(2, \mathbb{R})$. The foliated Laplacian $\Delta$ defines an essentially self adjoint operator on $L^2_0(SO(2, \mathbb{R}) \setminus X, \mu)$. The result of [AG13] concerns the spectrum $\Sigma_\mu$ of this operator.

**Theorem 5**[Avila-Gouëzel 13] *With the above notations, for any $\delta > 0$, $\Sigma_\mu \cap (0, 1/4 - \delta)$ is made of finitely many eigenvalues of finite multiplicities.*
Other formulation: *The essential spectrum of $\Delta$ on $L^2_0(SO(2, \mathbb{R})\backslash X, \mu)$ is contained in $[1/4, +\infty)$.*

Remarks

For the Masur-Veech Lebesgue measure on $SO(2, \mathbb{R})\backslash X$, it was known that 0 is isolated in the spectrum ([A-G-Yoccoz 06]).

The theorem was shown under the hypothesis that $\mu$ is related to the affine structure of $X$. A famous result of [Eskin-Mirzakhani 18] shows that all $SL(2, \mathbb{R})$-invariant ergodic probability measures on $X$ satisfy these algebraic conditions. In particular:
• \( \mu \) is invariant by the diagonal Teichmüller flow,

• \( \mu \) is locally the product of the Lebesgue measure on the orbits of the Teichmüller flow and the Lebesgue measures on affine subspaces of its stable and unstable manifolds,

• the Teichmüller flow uniformly expands and contracts the conditional measures on (un)stable manifolds.

The dynamics of the Teichmüller flow is nice, familiar and friendly in the compact parts of \( X \), but one has to control the excursions near infinity.
Some steps of the proof:

Control the essential spectrum of some resolvant of the Teichmüller flow on some space of distributions (à la Gouëzel-Liverani, see D. Dolgopyat’s talk).

Relate the eigenvalues of that resolvant to the poles of a meromorphic extension of the Laplace transforms of the correlation functions.

Do a reverse Ratner argument: describe the representations of $SL(2, \mathbb{R})$ compatible with these properties of the correlation functions.

Read the eigenvalues on the corresponding representations.
Most of the difficulties lie in the first step:

Let $D$ be the space of $C^\infty$ functions on $X$ with compact support and let $\mathcal{L}$ be the operator

$$\mathcal{L}f := \int_0^\infty e^{-4\delta t} f \circ \varphi_t \, dt.$$ 

**Theorem 6**[AG13] There exists two norms $\| \cdot \|, \| \cdot \|^\prime$ on $D$ such that
1. there is $C$ s. t., for $f \in D, t \geq 0$, $\| f \circ \varphi_t \| \leq C \| f \|,$
2. the unit ball in the $\| \cdot \|$ completion is relatively compact in the $\| \cdot \|^\prime$ norm and
3. there is a Doeblin-Fortet inequality: there exists $n$ such that

$$\| \mathcal{L}^n f \| \leq (1 + \delta)^n \| f \| + \| f \|^\prime.$$
It follows from Theorem 6 that the essential spectral radius of $\mathcal{L}$ on the $\|\cdot\|$ closure of $D$ is smaller than $(1 + \delta)$ (Hennion 93).

Observe that $\mathcal{L}1 = 1/4\delta$, so the spectral gap between the spectral radius and the essential spectral radius is huge.

Let $\lambda_i, i = 1, \cdots, \lambda_\ell$, be the eigenvalues of the operator $\mathcal{L}$ of modulus larger than $(1 + \delta)$, $\Pi_j$ the finite dimensional associated projectors.
Let $f_1, f_2 \in D$. The Laplace transform of the correlation function $F(z) := \int_0^\infty e^{-zt} \left( \int_X f_1 \circ \varphi_t \cdot d\mu \right) \, dt$ is holomorphic on $\Re z > 0$.

**Proposition** The function $F(z)$ has a meromorphic extension to the union of $\Re z > 0$ and a neighborhood of size $\delta$ of the segment $[-1 + 8\delta, 0]$. Its poles are at $4\delta - 1/\lambda_i, i = 1, \ldots, \ell$. Moreover, the residue at the pole $4\delta - 1/\lambda_i$ is $\int_X f_1 \prod_i f_2 \, d\mu$. 
Ingredients for the proof of Theorem 6:

- A good recurrence Finsler metric adapted to the dynamics of the Teichmüller flow (comes from [AGY 06]).
- An exponential recurrence estimate that control the return of the Teichmüller orbits in the compact part ([Eskin-Masur], [Athreya]).
- Gouëzel-Liverani norms involving an arbitrarily large number of derivatives (to get a large spectral gap).