Work of Sébastien Gouëzel on Limit Theorems and on Banach spaces adapted to dynamical systems
Plan of the talk

1. Background
2. Limit Theorems
3. Banach spaces
Deterministic systems can exhibit stochastic behavior in the sense that ergodic sums $A_N(x) = \sum_{n=0}^{N-1} A(f^n x)$ can satisfy the same limit theorems as sums of independent identically distributed random variables.

Observed in many systems, proven for a few.

Work of Sébastien Gouëzel
Review of limit theorems.

\[ S_N = \sum_{n=0}^{N-1} X_n, \ X_n \text{-i.i.d.} \]

- **Central Limit Theorem:** if \( X \in L^2 \) then
  \[
  \mathbb{P} \left( \frac{S_N - N\mathbb{E}(X)}{\sqrt{V(X)N}} \leq z \right) = \Psi(z) := \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du.
  \]

- **Local Limit Theorem:** \( \lim_{N \to \infty} \sqrt{N} \mathbb{P}(S_N \in z\sqrt{N} + I) = \frac{|I|}{\sqrt{2\pi}} e^{-z^2/2} \)
  unless \( \exists a, h : X \in a + h\mathbb{Z} \).

- **Berry–Esseen Theorem:** if \( X \in L^3 \) then
  \[
  \mathbb{P} \left( \frac{S_N - N\mathbb{E}(X)}{\sqrt{V(X)N}} \leq z \right) - \Psi(z) = O \left( \frac{1}{\sqrt{N}} \right).
  \]

- Normalizations \( \mathbb{E}(X), \ V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \) depend smoothly on \( X \).
Review of limit theorems.

\[ S_N = \sum_{n=0}^{N-1} X_n, \; X_n \text{-i.i.d.} \]

- **Central Limit Theorem**: if \( X \in L^2 \) then

\[
\mathbb{P} \left( \frac{S_N - N \mathbb{E}(X)}{\sqrt{V(X)N}} \leq z \right) = \psi(z) := \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.
\]

- **Stable laws**: If \( \mathbb{P}(X > t) = \frac{\ell_+(t)}{ts} \), \( \mathbb{P}(X < -t) = \frac{\ell_-(t)}{ts} \) where \( 0 < s < 2 \), and \( \ell_{\pm} \) are slowly varying, then \( S_N \) converges after proper normalization to stable laws.
Proofs of the above results use characteristic functions \( \Phi_X(\xi) = \mathbb{E} \left( e^{i\xi X} \right) \).

\[
\Phi \left( S_{N-N\mathbb{E}(X)} \frac{X - \mathbb{E}(X)}{\sqrt{\mathbb{V}(X)N}} \right) = \Phi \left( N \frac{X - \mathbb{E}(X)}{\sqrt{\mathbb{V}(X)N}} \right) \\
= \left( 1 - \frac{\xi^2}{2N} + \ldots \right)^N = e^{-\xi^2/2}(1 + \ldots).
\]
Spectral method (Nagaev–Guivarch)

$x_n$ Markov chain, $S_N = \sum_{n=0}^{N-1} A(x_n)$.

$$\Phi_N(\xi) = \mathbb{E}_{x_0} \left( e^{i\xi S_N} \right) = \mathbb{E} \left( e^{i\xi A(x_0)} \mathbb{E}_{x_1} \left( e^{i\xi S_{N-1}} \right) \right).$$

Let $(U_\xi B)(x) = \mathbb{E}_x \left( e^{i\xi A(x_0)} B(x_1) \right)$. Then

$$\Phi_N(\xi) = \langle U_N^\xi \mathbb{1}, \mathbb{1} \rangle = \langle \mathbb{1}, L_N^\xi \mathbb{1} \rangle \quad \text{where} \quad L_\xi = U_\xi^*.$$

Also $\mathbb{E}(B_1(x_0)B_2(x_N)) = \langle B_1, U_0^N B_2 \rangle = \langle L_0^N B_1, B_2 \rangle$.

**Example.** $P(x_0, dx_1) = k(x_0, x_1) d\nu(x_1)$. Then $U_\xi$ and $L_\xi$ are compact positive operators. Thus

$$U_\xi(A) = \lambda_\xi l_\xi(A) h_\xi + R_\xi, \quad \rho(R) \leq \theta < 1.$$ 

Thus $\Phi_N(\xi) = U_N^\xi(A) = \lambda_N^\xi l_\xi(A) h_\xi + O(\theta^N)$ and the asymptotic analysis proceeds as before.
Background
Inducing.
Banach spaces.

Spectral method (Nagaev–Guivarch)

$$\Phi_N(\xi) = \langle U_N^\xi, 1 \rangle = \langle 1, L_N^\xi 1 \rangle.$$ 

In fact compactness of $\mathcal{U}, \mathcal{L}$ is not necessary. It suffices that that $\mathcal{U}$ is quasi-compact, that is $\rho_{\text{ess}}(\mathcal{U}) < \rho(\mathcal{U}) = 1$.

Example (Doeblin). $P = q\hat{P} + (1 - q)\tilde{P}$ where $\hat{P}(x_0, dx_1) = \hat{k}(x_0, x_1)d\nu(x_1)$.

Way to prove quasi-compactness: Doeblin-Fortet (Lasota-Yorke) inequality

- $\mathcal{B} \subset \mathcal{B}_w$ $i: \mathcal{B} \to \mathcal{B}_w$ compact.
- $\exists, n, R$ such that

$$\|L^n(h)\| \leq R\|h\|_w + r^n\|h\| \Rightarrow \rho_{\text{ess}}(\mathcal{L}) \leq r.$$ 

For Markov chains: $\mathcal{B} = C^0, \mathcal{B}_w = L^1, r = (1 - q)$.
Lasota-Yorke inequality for expanding maps.

\( f : \mathbb{T} \to \mathbb{T} \) expanding, \( f'(y) \geq \Lambda > 1 \).

\[
(\mathcal{L} h)(x) = \sum_{f_y = x} \frac{h(y)}{f'(y)}
\]

\( \mathcal{B} = C^1, \mathcal{B}_w = C^0 \)

\[
| (D\mathcal{L} h)(x) | = \left| \sum_{f_y = x} \frac{Dh(y)}{(f'(y))^2} + \sum_{f_y = x} h(y) D \left( \frac{1}{f'(y)} \right) \right| \\
\leq \frac{1}{\Lambda} |\mathcal{L}(Dh)| + O \left( \|h\|_{C^0} \right).
\]

Hence \( \rho_{ess}(\mathcal{L}) \leq \frac{1}{\Lambda} \).

**Remark.** Taking \( \mathcal{B} = C^r, \mathcal{B}_w = C^{r-1} \) we get

\( \rho_{ess}(\mathcal{L}) \leq \Lambda^{-r} \).
Before (25 years ago):
Uniformly hyperbolic systems: expanding and piecewise expanding maps, subshifts of finite type.
Now: large class of non-uniformly hyperbolic systems and some partially hyperbolic systems.

Methods to go beyond uniform hyperbolicity:
(1) Inducing.
(2) Adapted Banach spaces.
Let $Y \subset X$ and $F : Y \to Y$ be given by $F(x) = f^{\tau(x)}(x)$. Then $f$ is naturally represented as tower over $F$.

Young (1998): Spectral approach works if

$$m(x : \tau(x) > n) \leq C \theta^n.$$

The case of polynomial tails remained open. Inducing is also often used in probability there it is a subject of renewal theory.
Renewal equation

\( f : X \to X \) tower over \( F : Y \to Y \). \( A, B \) are supported on \( Y \),

\[ \rho_n(A, B) = \langle A, B \circ f^n \rangle. \]

For \( |z| < 1 \) let

\[ t(z) = \sum_{n=0}^{\infty} \rho_n(A, B)z^n = \langle A, U_z(B) \rangle \]

where \( U_zB = \sum_n z^n B \circ f^n = B + z^T U_z(B \circ F) \). Thus

\[ t(z) = \langle T_zA, B \rangle = \langle A, B \rangle + < T_z(\mathcal{R}_zA), B \rangle. \]

Hence

\[ T_z = (I - \mathcal{R}_z)^{-1}. \]
Operator Renewal Theorem (Sarig-Gouëzel)

\[ T(z) = I + \sum_{n} z^n T_n, \quad R(z) = \sum_{n} z^n R_n : B \to B \text{ in } D = \{ |z| < 1 \}. \]

Assume

(a) **Renewal equation**: \( T(z) = (I - R(z))^{-1} \) in \( D \).

(b) **Spectral gap**: 1 is a simple isolated eigenvalue of \( R(1) \) on unit circle, with eigenprojection \( P \).

(c) **Aperiodicity**: \( I - R(z) \) is invertible on \( \bar{D} - \{1\} \).

(d) **Polynomial tails**: \( \sum_{k>n} R_k = O(1/n^\beta), \beta > 1. \)

Then \( T_n = \frac{1}{\mu} P + \frac{1}{\mu^2} \sum_{k>n} P_k + E_n \) where \( PR'(1)P = \mu(P) \),

\[ P_n = \sum_{l>n} PR_l P, \text{ and } \|E_n\| = O(\gamma_\beta(n)) \text{ where } \gamma_\beta(n) = 1/n^\beta \text{ if } \beta > 2, \quad \gamma_\beta(n) = 1/n^{2\beta - 2} \text{ if } 2 > \beta > 1. \]
Operator Renewal Theorem (Sarig-Gouëzel)

\[ T_n = \frac{1}{\mu} P + \frac{1}{\mu^2} \sum_{k>n} P_k + E_n \]

where \( PR'(1)P = \mu(P), \) \( P_n = \sum_{l>n} PR_l P. \)

**Corollary.** If \( PA = 0 \) then \( T_n A = O(1/n^\beta). \)

**Theorem.** If \( F \) is uniformly hyperbolic, \( A, B \) supported on \( Y \) then

(a) \[ \rho_N(A, B) - \mu(A)\mu(B) = \mu(A)\mu(B) \sum_{k>n} m(\tau > k) + O(\gamma_\beta(n)). \]

(b) If \( \mu(A) = 0 \) then \( \rho_n(A, B) = O(1/n^\beta). \)

(c) Without assuming that \( A, B \) are supported on \( Y \) we get

\[ \rho_N(A, B) - \mu(A)\mu(B) = O(1/n^{\beta-1}). \]

**Theorem.** Ergodic sums \( A_N(x) \) satisfy CLT if either \( \beta > 2 \) or \( \beta > 1, A \) is supported on \( Y \) and \( \mu(A) = 0. \)
Stadium and cusp

Let \( I_B = \int_{-\pi/2}^{\pi/2} A(s,0)ds, \ I_C = \int_{-\pi/2}^{\pi/2} [A(p_-, \phi) + A(p_+, \phi)]d\phi. \)

Assume \( \mu(A) = 0. \)

**Theorem.**

(a) Bálint-Gouëzel: For stadium \( \frac{A_N}{\sqrt{N \ln N}} \Rightarrow \mathcal{N}(0, cI_B^2). \) If \( I_B = 0 \), then standard CLT holds.

(b) Bálint-Chernov-Dolgopyat: the same result holds for cusp with \( I_B \to I_C. \)
Limit theorems by inducing.

Sébastien also demonstrated that inducing can be used to prove

- stable laws (2004);
- local limit theorem (2004);
- Berry–Esseen theorem (with rate $n^{-\delta/2}$ where $\delta = \min(\beta - 2, 1)$) (2004);
- almost sure CLT (Chazottes–Gouëzel (2007));
- almost sure invariance principle (2010);
- (polynomial) large and moderate deviations (Dedecker–Gouëzel–Merlevède (2018)).
Problems with inducing.

(I) If $F = f^\tau$ is uniformly hyperbolic and $\mathcal{F} = F^\sigma$ admits a Young tower then it may be difficult to estimate the tail of $\sigma(f^\tau(x)(x))$.

(II) Nearby systems have different symbolic description making it difficult to control dependence on parameters.

(III) $\rho_{ess}(\mathcal{L})$ for symbolic systems is non zero, making complete asymptotic expansions of correlation functions impossible.
Problems with inducing.

(I) If $F = f^\tau$ is uniformly hyperbolic and $\mathcal{F} = F^\sigma$ admits a Young tower then it may be difficult to estimate the tail of $\sigma(f^\tau(x))(x))$.

(II) Nearby systems have different symbolic description making it difficult to control dependence on parameters.

(III) $\rho_{ess}(\mathcal{L})$ for symbolic systems is non zero, making complete asymptotic expansions of correlation functions impossible.

**Alternative approach:** construct Banach spaces adapted to geometry of the system where transfer operator has good spectral properties.
Methods of designed adapted Banach spaces

(I) Geometric method. Contractions improve regularity of smooth functions.

\[ D^2(h(x)A(fx)) = h(x)B(fx)(f'(x))^2 + h_1 \int B(y)dy + h_2 \int \int B(y)dy \]

where \( B = A'' \). Consequently expanding maps improve regularity of distributions.

Transfer operators are associated to \( f^{-1} \). Thus one would like to work with functions which are smooth in the expanded directions and distributions in the contracted directions.

(II) Microlocal analysis. Example: \( L \)-linear Anosov map of \( \mathbb{T}^2 \).

\[ e_k(x) = \exp(2\pi i \langle k, x \rangle). \quad e_k(Lx) = e_{L^*k}(x). \]

The orbits of \( k \to (L^*)^nk \) are hyperbolic, so it is easy to construct Lyapunov function.
First attempt.

Take \( \beta < \gamma < \tau \)-regularity exponent of \( E^u, E^s \). 

\[
\|A\|_s = \sup_{\|\phi\|_{C^\beta_s} \leq 1} \int A\phi \, dm, \quad \|A\|_u = \sup_{\|v\|_{C^\beta_s} \leq 1, \, v \in E^u} v(A) \, dm
\]

\[
\|A\| = \|A\|_s + \|A\|_u, \quad \|A\|_w = \sup_{\|\phi\|_{C^\gamma_s} \leq 1} \int A\phi \, dm.
\]

Theorem. \( \rho_{ess}(\mathcal{L}) \leq \max \left( \frac{1}{\lambda^u}, \lambda^s \right) + \varepsilon. \)
First attempt.

Take $\beta < \gamma < \tau$-regularity exponent of $E^u, E^s$.

\[
\|A\|_s = \sup_{\|\phi\|_{C^s_\beta} \leq 1} \int A\phi dm, \quad \|A\|_u = \sup_{\|v\|_{C^s_\beta} \leq 1, v \in E^u} v(A) dm
\]

\[
\|A\| = \|A\|_s + \|A\|_u, \quad \|A\|_w = \sup_{\|\phi\|_{C^\gamma \tau} \leq 1} \int A\phi dm.
\]

**Theorem.** $\rho_{ess}(\mathcal{L}) \leq \max \left(\frac{1}{\lambda_u}, \lambda_s^\tau\right) + \varepsilon$.

Proved that one can get spectral gap without using symbolic dynamics.
First attempt.

Take $\beta < \gamma < \tau$-regularity exponent of $E^u, E^s$.

$$\|A\|_s = \sup_{\|\phi\|_{c^\beta_s} \leq 1} \int A\phi dm, \quad \|A\|_u = \sup_{\|v\|_{c^\gamma_s} \leq 1, v \in E^u} v(A) dm$$

$$\|A\| = \|A\|_s + \|A\|_u, \quad \|A\|_w = \sup_{\|\phi\|_{c^\gamma_s} \leq 1} \int A\phi dm.$$

**Theorem.** $\rho_{\text{ess}}(L) \leq \max \left( \frac{1}{\lambda_u}, \lambda_s^\tau \right) + \varepsilon$.

Proved that one can get spectral gap **without** using symbolic dynamics.

Did not solve the previous problems ((I) piecewise smoothness, (II)-parameter dependence, (III)-large essential spectrum).
Good Banach spaces.

(I) Can work with piecewise smooth maps.
(II) Are adapted to perturbations of $f$.
(III) Small essential spectrum.
Problems (II) and (III) were solved by Gouëzel-Liverani and by Baladi-Tsujii.
Problem (I) was addressed by Gouëzel-Baladi, Demers-Liverani, Demers-Zhang, Baladi-Demers . . . .

\[ \|A\|_{p,q}^-= \sup_W \sup_{v_1,v_2,\ldots,v_r} \sup_{\phi} \int_W (v_1 \ldots v_p)(A)\phi \, dm. \]

Here \( TW \in C_s, \phi \in C^q_0(W) \).

\[ \|A\|_{p,q} = \max_{0 \leq k \leq p} \|A\|_{k,q+k}. \]

**Theorem.** \( \rho_{\text{ess}} \leq \max(\lambda_u^{-p}, \lambda_s^q) + \varepsilon. \)

**Corollary.** \( \int A(x)B(f^n x) \, d\text{Vol}(x) \) admits complete asymptotic expansion.

**Theorem.** \( (A, f) \rightarrow \mu_{\text{SRB}}(A, f), (A, f) \rightarrow \sigma_{\text{SRB}}(A, f) \) are smooth.
Lasota–Yorke inequality

\[ \int_W (v_1 \ldots v_p)(\mathcal{L}^N A)\phi dm \]

If \( E^u \) is smooth we can

- Split \( v_j = v_j^u + v_j^W \);
- Transfer \( v_j^W \) derivatives on \( \phi \) by integrating by parts (controlling the result by \( \| \cdot \|_{p-k,q-k} \));
- Rewrite

\[ \int_W (v_1^u \ldots v_p^u)(\mathcal{L}^N A)\phi dm = \sum_{\alpha} \int_{W_\alpha} (Df^{-n}v_1^u) \ldots (Df^{-n}v_p^u)A\rho_j dm \]

gaining \( \lambda_{-\rho}^p \).
Lasota–Yorke inequality

If $E^u$ is not smooth, one can recover the same argument using Lemma. $\exists C$, such that for all $n$ if $\|v\| \leq 1$ then $v = \tilde{v}^W + \tilde{v}^u$ so that

$$
\|\tilde{v}^W\| \leq C, \quad \|Df^{-n}\tilde{v}^u\| \leq C\lambda^{-n}
$$

$$
\|Df^{-n}\tilde{v}^W\| \leq C_n, \quad \|\tilde{v}^u\| \leq C_n.
$$
Gouëzel-Liverani (2008)

Theorem (GL06).

\[ \int A(x)B(f^n x) d\text{Vol}(x) \quad \text{and} \quad \int A(x)B(f^n x) d\mu_{SRB} \]

admit complete asymptotic expansion.

Theorem (GL06). \((A, f) \to \mu_{SRB}(A, f), (A, f) \to \sigma_{SRB}(A, f)\) are smooth.

Gouëzel-Liverani (2008) obtain the same results with \(\mu_{SRB}\) replaced by Gibbs measures \(\mu_{\phi}\). Replacing

\[ \mathcal{L}(A)(x) = \frac{A(f^{-1} x)}{\det(df)(f^{-1} x)} \quad \text{by} \quad \mathcal{L}_\psi(A)(x) = \psi(f^{-1} x) \frac{A(f^{-1} x)}{\det(df)(f^{-1} x)} \]

where \(\psi(x) = \hat{\psi}(x, E^s(x))\) for a smooth function \(\hat{\psi}\) on \(\mathbb{P}(TM)\).
Selected recent applications of adapted Banach spaces

- Exponential mixing for Sinai billiard flows (Baladi–Demers–Liverani);
- order of vanishing at zero of Policott-Ruelle zeta function for geodesic flow on negatively curved surfaces (Dyatlov-Zworski);
- Ruelle spectrum for linear pseudo-Anosov maps and classification of invariant distributions for associated translation flows (Faure–Gouëzel–Lanneau);
- local marked spectrum rigidity for metrics with Anosov geodesic flows (Guillarmou–Lefeuvre).