## Schedule of talks
Semi-annual Workshop in Dynamical Systems and Related Topics
Pennsylvania State University, Oct. 27 - 30, 2016

**THURSDAY, October 27**

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<th>Time</th>
<th>Activity</th>
<th>Speaker</th>
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<td>1:00</td>
<td>Registration</td>
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<tr>
<td>1:15</td>
<td>Opening Remarks</td>
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<tr>
<td>1:20</td>
<td>Keith Burns</td>
<td>TBA</td>
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<tr>
<td>2:15</td>
<td>Kurt Vinhage</td>
<td>Smooth K non-Bernoulli examples in dimension 4</td>
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<td>3:05</td>
<td>Departmental Tea</td>
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<tr>
<td>3:35</td>
<td>Department of Mathematics Colloquium</td>
<td>Yuval Peres</td>
<td>Poisson boundaries and the Kaimanovich-Vershik conjecture</td>
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<tr>
<td>4:35</td>
<td>Alex Blumenthal</td>
<td>Lyapunov exponents for random perturbations of certain two-dimensional predominantly hyperbolic maps, including the standard map</td>
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<tr>
<td>5:30</td>
<td>Peter Nandori</td>
<td>Local limit theorem and mixing for certain hyperbolic flows</td>
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FRIDAY, October 28

9:00 - 9:50  Anton Gorodetski  
*Lyapunov exponents of random matrix products with parameter*

9:50 - 10:10  Coffee break

10:10 - 11:00  Sarah Bray  
*Ergodic geometry for nonstrictly convex Hilbert geometries*

11:10 - 12:00  Ilya Khayutin  
*Effective equidistribution and Hecke operators in dynamical systems*

12:00 - 1:30  Lunch break

1:30 - 2:20  Todd Fisher  
*Entropy of $C^1$ diffeomorphisms without a dominated splitting*

**Short talk sessions:**

2:30 - 2:55  Vladimir Finkelshtein (113)  
*Diophantine approximation problems for groups of toral automorphisms*

3:00 - 3:25  Changguang Dong (114)  
*Separated nets arising from higher rank $\mathbb{R}^k$ actions on certain homogeneous spaces*

3:30 - 3:50  Coffee break

4:00 - 4:25  Jianyu Chen (113)  
*Statistical properties of one-dimensional expanding maps with singularities of low regularity*

4:30 - 4:55  Yuki Takahashi (113)  
*An error term in the Central Limit Theorem for sums of discrete random variables*

5:00 - 5:50  Svetlana Jitomirskaya  
*Universal hierarchical structure of eigenfunctions and non-regular dynamics of quasiperiodic Schrödinger cocycles*

7:00 -  7:00  Banquet at Fuji & Jade Garden Restaurant  
*418 Westerly Pkwy, State College, PA 16801*
### SATURDAY, October 29

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<thead>
<tr>
<th>Time</th>
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<th>Topic</th>
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<tr>
<td>9:00 - 9:50</td>
<td>Joe Auslander</td>
<td>Regional proximality and the Veech relation in minimal flows</td>
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<tr>
<td>9:50 - 10:10</td>
<td>Coffee break</td>
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<tr>
<td>10:10 - 11:00</td>
<td>Alex Gamburd</td>
<td>Markoff Surfaces and Strong Approximation</td>
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<tr>
<td>11:10 - 12:00</td>
<td>Vaughn Climenhaga</td>
<td>A direct proof of the entropy gap for rank 1 manifolds</td>
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<td>12:00 - 1:30</td>
<td>Lunch break</td>
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<tr>
<td>1:30 - 2:20</td>
<td>Boris Hasselblatt</td>
<td>Desingularization of singular-hyperbolic systems</td>
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**Session on Dynamical Systems Prize for Young Mathematicians:**

<table>
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<th>Time</th>
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<td>Prize Ceremony</td>
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<tr>
<td>2:50 - 3:40</td>
<td>Talk 1</td>
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<td>3:40 - 4:00</td>
<td>Coffee break</td>
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<td>4:00 - 4:50</td>
<td>Talk 2</td>
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<td>5:00 - 5:50</td>
<td>Talk 3</td>
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<td>Time</td>
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<td>9:00 - 9:50</td>
<td>Konstantin Khanin</td>
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<td>10:00 - 10:50</td>
<td>Aaron Brown</td>
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<td>11:00 - 12:00</td>
<td>Dan Thompson</td>
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<tr>
<td>12:10 - 1:00</td>
<td>Omri Sarig</td>
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Joe Auslander (University of Maryland)

**Regional proximality and the Veech relation in minimal flows**

Let \((X,T)\) be a flow, with \(X\) compact Hausdorff. A classical theorem of Gottschalk and Ellis is that the equicontinuous structure relation \(S_{eq}\) is generated by \(RP\), the regionally proximal relation. If \((X,T)\) is minimal, then in many cases (including \(T\) abelian and \((X,T)\) distal) \(RP\) is an equivalence relation, so \(S_{eq} = RP\). A remarkable result of Veech is that if \(T\) is abelian and \(X\) is a metric space, the apparently simpler relation \(V\), defined by \((x,y) \in V\) if there is a \(z \in X\) and a sequence \(t_n \in T\) with \(t_n x \to z\) and \(t_n^{-1} z \to y\) is in fact equal to \(S_{eq}\).

We will present an alternate proof and generalization of Veech’s theorem, in collaboration with Anima Nagar and Gernot Greschonig. We will also discuss the connection of the Veech relation with the regional proximal relation of order \(d\) defined by Host, Kra, and Maass.

Alex Blumenthal (University of Maryland)

**Lyapunov exponents for small random perturbations of predominantly hyperbolic two-dimensional volume-preserving diffeomorphisms, including the Standard Map**

An outstanding problem in smooth ergodic theory is the estimation from below of Lyapunov exponents for maps which exhibit hyperbolicity on a large but non invariant subset of phase space. It is notoriously difficult to show that Lyapunov exponents actually reflect the predominant hyperbolicity in the system, due to cancellations caused by the ‘switching’ of stable and unstable directions in those parts of phase space where hyperbolicity is violated.

In this talk I will discuss the inherent difficulties of the above problem, and will discuss recent results when small IID random perturbations are introduced at every time-step. In this case, we are able to show with relative ease that for a large class of volume-preserving predominantly hyperbolic systems in two dimensions, the top Lyapunov exponent actually reflects the predominant hyperbolicity in the system. Our results extend to the well-studied Chirikov Standard Map at large coupling. This work is joint with Lai-Sang Young and Jinxin Xue.

Sarah Bray (University of Michigan)

**Ergodic geometry for nonstrictly convex Hilbert geometries**

Strictly convex Hilbert geometries naturally generalize constant negatively curved Riemannian geometries, and the geodesic flow on quotients has been well-studied by Benoist, Crampon, Marquis, and others. In contrast, nonstrictly convex Hilbert geometries in three dimensions have the feel of nonpositive curvature, but also have a fascinating geometric irregularity which forces the geodesic flow to avoid direct application of existing nonuniformly hyperbolic theory. In this talk, we present our approach to studying the geodesic flow in this setting, culminating in a measure of maximal entropy which is ergodic for the geodesic flow.

Aaron Brown (University of Chicago)

**Zimmers conjecture for cocompact lattices**

D. Fisher, S. Hurtado, and I recently established the following version of Zimmers conjecture: For \(n \geq 3\), consider a cocompact lattice in \(SL(n, \mathbb{R})\) acting on a manifold \(M\). If \(\dim(M) < n - 1\) then the action is finite; if \(\dim(M) = n - 1\) and the action preserves a volume then the action is finite. The proof uses tools
from homogeneous dynamics, cocycle superrigidity, Strong Property (T), and smooth ergodic theory for actions of abelian groups. I will give an outline of the proof and state a key proposition based on earlier work of myself, F Rodriguez Hertz, and Z. Wang.

Keith Burns (Northwestern University)
TBA

Dong Chen (Pennsylvania State University)
Two types of KAM-nondegenerate nearly integrable systems with positive metric entropy

The celebrated KAM Theory says that if one makes a small perturbation of a non-degenerate completely integrable system, we still have a huge measure of invariant tori with quasi-periodic dynamics in the perturbed system. These invariant tori are known as KAM tori. What happens outside KAM tori draws lots of attention. In this talk I will present two types of $C^\infty$ small Lagrangian perturbation of the geodesic flow on a flat torus. Both resulting flows have positive metric entropy, from which we get positive metric entropy outside some KAM tori. What is special in the second type is that positive metric entropy comes from an arbitrarily small tubular neighborhood of one trajectory. This is a joint work with D. Burago and S. Ivanov.

Jianyu Chen (University of Massachusetts, Amherst)
Statistical properties of one-dimensional expanding maps with singularities of low regularity

We consider one-dimensional non-Markov uniformly expanding maps, whose derivatives may blow up and have unbounded variations. Under certain conditions, we are able to show the existence of absolutely continuous invariant measure, exponential decay or correlations, central limit theorem and large deviation principle. Our method applies to $C^1$ perturbations of the tent map, the beta transformation, the Gauss map, etc.

Vaughn Climenhaga (University of Houston)
A direct proof of the entropy gap for rank 1 manifolds

The classical theory of thermodynamic formalism for uniformly hyperbolic systems shows that for geodesic flow over a compact Riemannian manifold of negative curvature, every Holder continuous potential function has a unique equilibrium state. For rank 1 manifolds of non-positive curvature, the set of singular geodesics (along which the stable and unstable horospheres have a higher-order tangency) acts as an obstruction to hyperbolicity, and the geodesic flow is non-uniformly hyperbolic. Knieper proved uniqueness of the measure of maximal entropy in this setting, and from that result was able to deduce that the entropy of the singular set is smaller than the entropy of the entire system.

I will present a direct proof of this entropy gap result using dynamical methods, which also yields the corresponding result for pressure whenever the potential is locally constant on a neighbourhood of the singular set. For certain rank 1 manifolds, this shows that the pressure gap holds for an open and dense set of potential functions. I will also describe how one can obtain a unique equilibrium state for any potential function satisfying the pressure gap condition, which in particular gives an alternate proof of Knieper’s result for the measure of maximal entropy. This is joint work with Keith Burns, Todd Fisher, and Dan Thompson.

Changguang Dong (Pennsylvania State University)
Separated nets arising from higher rank $\mathbb{R}^k$ actions on certain homogeneous spaces

A subset $N \subset \mathbb{R}^k$ is a separated net, if there exists two positive numbers $r, R$ ($r < R$), such that every
ball of radius \( r \) contains at most one point from \( N \) and every ball of radius \( R \) contains at least one point from \( N \). A simple example of separated net is a lattice of \( \mathbb{R}^k \). We say, two separated nets \( N_1 \) and \( N_2 \) are bi-Lipschitz equivalent (BL) if there is a bijection \( \phi : N_1 \to N_2 \) which is bi-Lipschitz, i.e. there exists \( C > 0 \), such that for any \( x, y \in N_1 \),

\[
C^{-1}||x - y|| \leq ||\phi(x) - \phi(y)|| \leq C||x - y||.
\]

In this talk, we show that separated net arising from certain higher rank \( \mathbb{R}^k \) actions on homogeneous spaces is bi-Lipschitz equivalent to a lattice.

**Kasun Fernando** (University of Maryland)

An error term in the Central Limit Theorem for sums of discrete random variables

We consider sums of independent identically distributed random variables whose distributions have \((d + 1), \ d > 1\) atoms. Such distributions never admit an Edgeworth expansion of order \( d \) but we show that for almost all parameters the Edgeworth expansion of order \( d-1 \) is valid and the error of the order \( d \) Edgeworth expansion is typically of order \( n^{-d/2} \). This is a joint work with Dmitry Dolgopyat.

**Vladmir Finkelshtein** (University of Illinois, Chicago)

Diophantine approximation problems for groups of toral automorphisms

I will present sharp rates for a shrinking target problem for the action of arbitrary subgroups of \( \text{SL}(2, \mathbb{Z}) \) on the 2-torus. This can also be viewed as a noncommutative Diophantine approximation problem. The methods require construction of spectrally optimal random walks on groups acting properly cocompactly on Gromov hyperbolic spaces. Additionally, I will explain how similar estimates for this problem in higher dimension can be obtained using harmonic analysis.

**Todd Fisher** (Brigham Young University)

Entropy of \( C^1 \) diffeomorphisms without a dominated splitting

A classical construction due to Newhouse creates horseshoes from hyperbolic periodic orbits with large period and weak domination through local perturbations. Our main theorem shows that, when one works in the \( C^1 \) topology, the entropy of such horseshoes can be made arbitrarily close to an upper bound deriving from Ruelle’s inequality, i.e., the sum of the positive Lyapunov exponents (or the same for the inverse diffeomorphism, whichever is smaller). Adapting classical techniques, we use perturbations that are local and can be chosen to preserve volume or symplectic form or a homoclinic connection.

This optimal entropy creation yields a number of consequences for \( C^1 \)-generic diffeomorphisms, especially in the absence of a dominated splitting. For instance, in the conservative settings, we find formulas for the topological entropy, deduce that the topological entropy is continuous but not locally constant at the generic diffeomorphism and we prove that these generic diffeomorphisms have no measure of maximum entropy. In the dissipative setting, we show the locally generic existence of infinitely many homoclinic classes with entropy bounded away from zero.

**Alexander Gamburd** (City University of New York)

Markoff Surfaces and Strong Approximation

**Anton Gorodetski** (University of California, Irvine)

Lyapunov exponents of random matrix products with parameter

Random products of matrices appear naturally in many different settings, in particular in smooth dynamical systems, probability theory, spectral theory, mathematical physics. The crucial result is Furstenberg’s
Theorem on positivity of Lyapunov exponents. It claims that generically the exponential rate of growth (Lyapunov exponent) of product of random matrices is well defined and positive. In the talk we will discuss the random products of $2 \times 2$ matrices that depend on a parameter. This is motivated, in particular, by the study of discrete Schrodinger operators with random potentials. In that case the Schrodinger cocycle is given by the random products of transfer matrices, and energy serves as a natural parameter. From spectral point of view it is natural to fix the potential first, and then vary the energy. As a more general setting, one can consider random products of matrices depending on a parameter, and study existence and properties of Lyapunov exponent for a typical fixed sequence when the parameter varies. We will show, for example, that in the non-uniformly hyperbolic regime almost surely upper Lyapunov exponent is positive (and coincides with the one prescribed by Furstenberg Theorem) for all parameters, but lower Lyapunov exponent vanishes for a topologically generic parameter. These result explain the difficulties one encounters in the classical proofs of Anderson localization for random Schrodinger operators. This is a joint project with V. Kleptsyn.

**Boris Hasselblatt** (Tufts University)

*Desingularization of singular-hyperbolic systems*

The study of the Palis conjectures led to investigations of singular hyperbolic attractors, and a construction by Bonatti, Pumarino and Viana answered the question of whether robust such attractors exist whose expading direction has dimension 2 or higher. This being the only known construction we consider the question of whether this is the only possible construction by solving a related toy problem for maps of surfaces.

**Svetlana Jitomirskaya** (University of California, Irvine)

*Universal hierarchical structure of eigenfunctions and non-regular dynamics of quasiperiodic Schrodinger cocycles*

For a cocycle over a transformation acting on a space $X$ (Lyapunov-Perron), non-regular points $x \in X$ are the ones at which Oseledets multiplicative ergodic theorem does not hold coherently in both directions. They therefore form a measure zero set with respect to any invariant measure on $X$. Yet, it is precisely the non-regular points that are of interest in the study of Schrodinger cocycles in the positive Lyapunov exponent regime, since for every $x$, the spectral measures are supported on energies such that $x$ is a non-regular point of the corresponding non-uniformly hyperbolic transfer-matrix cocycle.

We will present exact exponential asymptotics of all eigenfunctions and of corresponding cocycles for the almost Mathieu operators for all parameters in the localization regime. This uncovers a universal structure in their behavior. For the frequency resonances, there is a hierarchy governed by the continued fraction expansion of the frequency, explaining predictions in physics literature. For the phase resonances, we will describe a new phenomenon, not even previously described in physics: a reflexive hierarchy, where self-similarity holds upon alternating reflections.

In addition the asymptotics leads to a proof of the sharp arithmetic version of both frequency and phase transition conjectures. Finally, it leads to an explicit description of several non-regularity phenomena in the corresponding non-uniformly hyperbolic cocycles, which is also of interest as both the first natural example of some of those phenomena and, more generally, the first non-artificial model where non-regularity can be explicitly studied. the talk is based on papers joint with W. Liu.

**Konstantin Khanin** (University of Toronto)

*On typical irrational rotation numbers for circle maps with singularities*

We shall discuss an approach to define typical irrational rotation numbers for circle maps with singular-
Hecke Operators are ubiquitous in the theory of automorphic forms. We present a simple construction of averaging operators on state spaces of measurable dynamical systems. This is a common generalization of Hecke operators from number theory and Markov shifts. It is also closely related to the Laplacian on a Riemannian manifold.

Our first objective is showing an effective ergodic theorem with an exponential rate large deviations using a norm gap for the averaging operator. I will present a general criterion for states spaces of dynamical systems which implies a relatively sharp large deviations result. A large class of such systems arises from S-arithmetic quotients of reductive groups. This part builds upon the work of Kahale and Ellenberg, Michel and Venkatesh.

Last I will discuss how these methods imply an effective version of uniqueness of the measure of maximal entropy and non-escape of mass for sequences of measures with high entropy for the dynamical systems in question. The relation between large deviations and equidistribution was pioneered by Linnik and later studied by Ellenberg, Michel and Venkatesh.

Peter Nandori (University of Maryland)

Local limit theorem and mixing for certain hyperbolic flows

Let us consider a suspension (semi-)flow with roof function \( \tau \) over some map \( T : X \to X \). For both integrable and non-integrable \( \tau \), we try to give some abstract conditions, under which the flow is mixing. In case of integrable \( \tau \), some extra conditions on the observable imply a joint extension of mixing and the local central limit theorem (LCLT) for the flow. The most important condition is the LCLT for the map \( T \). Examples include certain flows derived from the Liverani-Saussol-Vaienti map (infinite measure case) and some suspensions over systems possessing a Young tower with exponential return time (finite measure case). Joint work in progress with Dmitry Dolgopyat.

Wenyu Pan (Yale University)

Joining Measures for horocycle flows on abelian covers

A celebrated result of Ratner from the eighties says that two horocycle flows on hyperbolic surfaces of finite area are either the same up to algebraic change of coordinates, or they have no non-trivial joinings. Recently, Mohammadi and Oh extended Ratner’s theorem to horocycle flows on hyperbolic surfaces of infinite area but finite genus. In this paper, I will present a joining classification result of a horocycle flow on a hyperbolic surface of infinite genus: a \( \mathbb{Z} \) or \( \mathbb{Z}^2 \)-cover of a general compact hyperbolic surface. I will also discuss several applications.

Yuval Peres (Microsoft Research)

Poisson boundaries and the Kaimanovich-Vershik conjecture

The crucial role of entropy in dynamical systems is well known; it is equally important in the study of random walks on groups. Kaimanovich and Vershik (1983) described the lamplighter groups (amenable groups of exponential growth consisting of finite lattice configurations) where (in dimension at least 3) simple random walk has positive asymptotic entropy and the Poisson boundary is nontrivial. They conjectured a
complete description of the Poisson boundary of these groups; In dimension 5 and above, their conjecture was proved by Anna Erschler (2011). I will discuss the background and present a proof of the Kaimanovich-Vershik conjecture for all dimensions, obtained in joint work with Russ Lyons. The proof uses a new, enhanced version of the classical Kaimanovich criterion for boundary maximality. The case of dimension 3 is the most delicate. http://arxiv.org/abs/1508.01845

Omri Sarig (Weizmann Institute of Science)
*Measures of maximal entropy for $C^\infty$ surface diffeomorphisms with positive entropy*

We show that a $C^\infty$ diffeomorphism with positive entropy of a compact surface has at most finitely many ergodic measures of maximal entropy, and in the topologically transitive case — exactly one. This is joint work with Jerome Buzzi and Sylvain Crovisier

Yuki Takahashi (University of California, Irvine)
*Sums and products of Cantor sets and separable two-dimensional quasicrystal models*

Arithmetic sums of two Cantor sets appear naturally in dynamical systems, number theory, and also spectral theory. Indeed, the spectrum of the tridiagonal square Fibonacci Hamiltonians, which is a two-dimensional quasicrystal model, is given by sums of two Cantor sets. We show the existence of an open set of parameters which yield mixed interval-Cantor spectra (i.e. spectra containing an interval as well as a Cantor set). On the other hand, the spectrum of the Labyrinth model, which is another two-dimensional quasicrystal model, is given by products of two Cantor sets. We give the optimal estimates in terms of thickness that guarantee that products of two Cantor sets is an interval, and apply this result to show that the spectrum of the Labyrinth model is an interval for sufficiently small coupling constants. We also consider sums of homogeneous Cantor sets, and show that for any two homogeneous Cantor sets with sum of Hausdorff dimensions exceeding 1, one can create an interval in the sumset by applying arbitrary small perturbations to the expanding maps (without leaving the class of homogeneous Cantor sets, but possibly refining the corresponding Markov partition).

Dan Thompson (Ohio State University)
*Thermodynamic formalism for geodesic flow on locally CAT(-1) spaces*

Locally CAT(-1) spaces are geodesic metric spaces satisfying a metric notion of negative curvature. These spaces are not necessarily manifolds, covering examples such as graphs equipped with an interior metric, yet they still have a geodesic flow defined on them. While there are analogies with the dynamics of the geodesic flow on a negative curvature Riemannian manifold, the full power of uniformly hyperbolic dynamics is not currently available in this setting: there is a coding of the geodesic flow by a suspension flow over a shift of finite type, but rather than being a conjugacy, it is via an orbit semi-equivalence.

In general, orbit equivalence is too weak a relationship to preserve any refined dynamical information. However, we are able to use a geometric argument to show that the geodesic flow inherits the weak specification property from the suspension flow. We use the specification property directly on the geodesic flow for CAT(-1) spaces, obtaining thermodynamic results analogous to the negative curvature Riemannian setting. In particular, we prove uniqueness of equilibrium states for Holder potential functions, the large deviations principle, and the equidistribution of weighted periodic orbits.

This is joint work with Dave Constantine and Jean-Francois Lafont.

Kurt Vinhage (University of Chicago)
*Smooth K non-Bernoulli examples in dimension 4*
The Kolmogorov (K) and Bernoulli properties are two fundamental notions in measurable dynamics. That Bernoulli shifts are K follows from the fact that factors of Bernoulli shifts remain Bernoulli. While it was shown by Ornstein that K does not imply Bernoulli, they are equivalent in certain settings (for instance, non-uniformly hyperbolic, volume-preserving diffeomorphisms). We exhibit, for the first time, diffeomorphisms of 4-manifolds which are K but not Bernoulli. These examples are also unique in that they are skew product systems with a loosely Bernoulli fiber. Joint with A. Kanigowski and F. Rodriguez-Hertz.

Agnieszka Zelerowicz (Pennsylvania State University)

Thermodynamics of some non-uniformly hyperbolic attractors

We study thermodynamical formalism for certain dissipative maps, that is maps with non-uniformly hyperbolic attractors, which are obtained from uniformly hyperbolic systems by the slow down procedure. Namely, starting with a hyperbolic local diffeomorphism $f : U \to M$ with an attractor $\Lambda$, one slows down trajectories in a small neighborhood of a hyperbolic fixed point $p \in \Lambda$ obtaining a nonuniformly hyperbolic diffeomorphism $g : U \to M$ with a topological attractor $\Lambda_g$. We establish the existence of equilibrium measures for any continuous potential function on $\Lambda_g$, however our main focus is the family of geometric $t$-potentials defined by $\varphi_t(x) := -t \log |Df|_{E^u}(x)|$. We prove the existence of $t_0 < 0$ such that the equilibrium measures are unique for every $t \neq 1$ that belongs to the interval $(t_0, \infty)$. We also identify equilibrium measures for $t = 1$. Finally, we show that for $t \in (t_0, 1)$ the equilibrium measures have exponential decay of correlations and satisfy the Central limit theorem.