QUALIFYING EXAMINATION IN REAL ANALYSIS
December 21, 2019

Instructions: To pass the exam you must correctly solve at least two of the following four problems.

Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Suppose that \( f : [0, 1] \to \mathbb{R} \) is Lebesgue measurable. Prove that there is a Borel measurable function \( g : [0, 1] \to \mathbb{R} \) such that \( f = g \) Lebesgue almost everywhere.

2. Let \( \langle f_n \rangle \) be a sequence of real-valued Lebesgue measurable functions on \([0, 1]\). Such a sequence is called Cauchy in measure if for all \( \varepsilon > 0 \), there is an \( N \) such that whenever \( n, k > N \),

\[
\lambda\{x : |f_n(x) - f_k(x)| > \varepsilon\} < \varepsilon.
\]

Prove that if \( \langle f_n \rangle \) is Cauchy in measure, then there is a Lebesgue measurable function \( f : [0, 1] \to \mathbb{R} \) such that \( f_n \) converges to \( f \) in measure.

3. Consider a Lebesgue measurable function \( f : [0, 1] \to [0, 1] \). Prove that there exists \( \omega \in [0, 1] \) such that

\[
\int_0^1 \frac{dx}{|f(x) - \omega|} \geq 2019.
\]

4. Let \( E \subseteq [0, 1] \) be a Lebesgue measurable set. Suppose \( f \in L^1([0, 1]) \) and for every \( x \in E \), \( f \) is differentiable at \( x \) and \( |f'(x)| \leq 1 \). Prove that \( f(E) \) is measurable, and \( \lambda(f(E)) \leq \lambda(E) \).