Instructions: To pass the exam you must correctly solve at least two of the following four problems.

Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Suppose \((F_n)_{n \in \mathbb{N}}\) are a sequence of closed sets with \(\bigcup_n F_n = \mathbb{R}\). Suppose that \(f : \mathbb{R} \to \mathbb{R}\) is measurable and \(f\) is Lebesgue integrable on \(F_n\) for each \(n\). Prove that \(\{x \in \mathbb{R} : f\) is Lebesgue integrable in some open neighborhood of \(x\}\) is dense in \(\mathbb{R}\).

2. Prove the following simplified version of uniform convexity of \(L^p\), \(p \in (1, \infty)\). For every \(\epsilon \in (0, 1)\) there exists \(\eta < 1\) such that for every pair of functions \(f, g \in L^p([0, 1])\), if \(\|f\|_p = \|g\|_p = 1\), and \(\|f\|_{\infty}, \|g\|_{\infty} \leq 2019\), and \(\|f - g\|_p > 2\epsilon\) then \(\|f + g\|_p \leq 2\eta\).

3. Let \(\mu\) be the Lebesgue measure on \(\mathbb{R}\). Suppose that \(f : \mathbb{R} \to \mathbb{R}\) is measurable, \(f(B)\) is measurable for every Borel set \(B \subseteq \mathbb{R}\), and \(\{y : f^{-1}(y)\) is infinite\} has Lebesgue measure 0. Suppose there is some \(A \subseteq \mathbb{R}\) such that \(\mu(A) = 0\) and \(\mu(f(A)) > 0\). Prove that there is a closed set \(F\) such that \(\mu(F) = 0\) and \(\mu(f(F)) > 0\).

4. Suppose \(f\) and its Hardy-Littlewood maximal function \(Mf\) are both in \(L^1(\mathbb{R}^2)\). Prove that \(f = 0\) almost everywhere. Reminder:

\[
Mf(x) = \sup_{r > 0} \frac{1}{\pi r^2} \int_{B(x,r)} |f(y)| \, dy.
\]