A brief survey of point set topology: open and closed sets, metric spaces, continuous maps and homeomorphisms, Hausdorff spaces, compactness. A collection of results: a compact subspace of a Hausdorff space is closed; a continuous image of a compact is compact; compacts in $\mathbb{R}^n$ are closed and bounded; a closed subset of a compact is compact; a continuous bijection $X \to Y$ with $X$ compact and $Y$ Hausdorff is a homeomorphism. Connectedness and path connectedness; a continuous image of a connected space is connected. Quotient topology.

Examples of spaces: $D^n$, $S^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, Grassmannians $G(k, n)$ and $G^+(k, n)$; Plücker coordinates. Example: $G^+(2, 4) \cong S^2 \times S^2$. Compact classical groups: $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$. Example: $SO(3) \cong \mathbb{RP}^3$. Hopf fibrations $S^3 \to S^2$ and $S^{2n+1} \to \mathbb{CP}^n$. Homogeneous spaces:

$$S^{n-1} = SO(n)/SO(n-1), S^{2n-1} = SU(n)/SU(n-1), G(k, n) = O(n)/O(k) \times O(n-k).$$

Classical closed surfaces, connected sums: $M_g = T^2 \# \ldots \# T^2$ and $\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2$.

Classification theorem (without proof). Polygonal models, orientation, and Euler characteristic.


CW (cell) complexes, skeletons. Cell structures of $S^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, classical surfaces. CW subcomplexes.

Operations on spaces and cell complexes: product, quotient space, suspension, join, wedge sum, and smash product. Examples: $S^p \times S^q = S^{p+q+1}, S^p \wedge S^q = S^{p+q}$.

Extensions of homotopy. Theorem: a CW pair possesses the extension of homotopy property. Corollary: if $(X, A)$ is a CW pair and $A$ is contractible, then $X \to X/A$ is a homotopy equivalence.

Fundamental group: definition, dependence on the fixed point. $\pi_1(S^1) = \mathbb{Z}$. Application: Brouwer fixed point theorem for 2-discs. Another application: winding number of an immersed plane curve. Still another application: Borsuk-Ulam theorem on $S^n \to \mathbb{R}^2$ and the sandwich theorem in $\mathbb{R}^3$. Proposition: $\pi_1(S^n) = 0$ for $n \geq 2$. Application: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$. Proposition: a homotopy equivalence induces an isomorphism of fundamental groups.

Finitely-generated groups: generators, relations. Van Kampen theorem. Fundamental groups of the classical surfaces. The effect of attaching a cell.
Fundamental groups of CW complexes.

Covering spaces. Examples: coverings of $S^1$, coverings of $S^1 \lor S^1$. Embedding of $F_3$ in $F_2$. Coverings $S^n \to \mathbb{R}P^n$, lens spaces.

Lifting properties: path lifting lemma and homotopy lifting lemma. The homomorphism of fundamental groups induced by a covering is injective. Bijection between $p^{-1}(x)$ and $\pi_1(X,x)/p_*\pi_1(\tilde{X},\tilde{x})$.

Deck transformations. Theorem: the group of deck transformations is isomorphic to $N(p_*\pi_1(\tilde{X},\tilde{x}))/p_*\pi_1(\tilde{X},\tilde{x})$. Regular coverings. Universal coverings.

Criterion for lifting a map $(Y,y) \to (X,x)$ to a covering space $(\tilde{X},\tilde{x}) \to (X,x)$. Equivalence of coverings. Criterion for two coverings to be equivalent. Existence of a covering with a given group and the hierarchy of coverings.

Further applications of the fundamental group: FTA, nonexistence of tangent vector fields on $S^2$, Cayley graphs and Cayley complexes.

Conway’s zip proof of the classification of classical surfaces.

Standard simplex, singular simplices. Singular chain, boundary operator. Lemma: $\partial^2 = 0$. Definition of homology: $H_n = \text{Ker} \partial_n/\text{Im} \partial_{n+1}$. First computations: $H_0(X)$, homology of a point.

Homotopy invariance of homology groups: prism construction.

Relative homology; long exact sequence.

Theorem: $H_n(X,A) = H_n(X/A)$. Applications: homology of spheres, Brouwer’s fixed point theorem, degree of a map $S^n \to S^n$. Theorem: $S^n$ carries a non-vanishing vector field iff $n$ is odd.

Proof of $H_n(X,A) = H_n(X/A)$. Singular chains subordinated to a covering. Barycentric subdivision and its properties. Application to the covering of $X \cup CA$ consisting of two sets. Consequence: the excision lemma.

Applications: invariance of dimension; Mayer-Vietoris sequence; homology of the wedge product of spaces; a local formula for the degrees of maps $S^n \to S^n$.

Homology of CW complexes: cellular chain complex; theorem: its homology coincide with the singular homology. Application: Euler characteristic can be calculated by any cell decomposition. Differentials of the cellular chain complex. Calculations: homology groups of $\mathbb{C}P^n$, $\mathbb{R}P^n$, and the classical surfaces.

Applications: the commutative division algebras are $\mathbb{R}$ and $\mathbb{C}$ (Hopf’s theorem); Borsuk-Ulam theorem and the exact homology sequence of a 2-fold covering.
The relation between $\pi_1(X)$ and $H_1(x)$: the latter is the Abelianization of the former.

Morse theory: non-degenerate critical points, Morse lemma, and Morse index. Counting polynomials for a Morse function and a manifold; Morse inequalities $P_f(t) = P_M(t) + (1 + t)Q(t)$, where $Q$ has non-negative coefficients. Sketch of the proof: how the topology of the sub-level set changes as one passes through a critical value. Application: the lower bound on the number of diameters of a convex hypersurface in $\mathbb{R}^n$.


Coefficient sequences in homology and cohomology, Bockstein homomorphism.

Closed manifolds and Poincaré duality. Fundamental class of a manifold. Intersection bilinear form $H_i(M^n) \times H_{n-1}(M^n) \to \mathbb{Z}$. Geometrical meaning: transverse intersection of submanifolds of complimentary dimensions. Examples: $M_g, \mathbb{R}P^n$.


Applications: a homeomorphism of $\mathbb{C}P^n$ with even $n$ is orientation preserving; Borsuk-Ulam theorem.