Instructions: To pass the exam you must correctly solve at least two of the following four problems.

Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Problem 1. Does there exist a $3 \times 3$ matrix with entries in $\mathbb{C}$ which commutes with

$$
A = \begin{pmatrix}
5 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

but is not of the form $f(A)$ for some polynomial $f \in \mathbb{C}[x]$?

Problem 2. Let $f(x)$ be the polynomial

$$
f(x) = (x^2 - 1)(x^2 + 1).
$$

Consider the set of real $3 \times 3$ matrices $A$ such that $\det A = 1$ and $f(A) = 0$. Determine the number of (real) similarity classes of such matrices, and give a representative of each class.

Problem 3. Consider the real quadratic form $Q : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$$
Q(x, y, z) = 4xy - 2xz - 2yz.
$$

Show that if $v_1, v_2 \in \mathbb{R}^3$ are any two linearly independent vectors, then there is a nontrivial linear combination $\alpha v_1 + \beta v_2 \neq 0$ such that

$$
Q(\alpha v_1 + \beta v_2) = -1.
$$

Problem 4. Assume that $A$ and $B$ are $n \times n$ real, symmetric, positive semidefinite matrices. Prove that

$$
\det(A + B) \geq \max\{\det A, \det B\}.
$$