ON OMRI SARIG’S WORK ON THE DYNAMICS ON SURFACES

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We present a short survey of two major achievements of Omri Sarig’s. Both concern low-dimensional dynamics, but with very different points of view: the first one constructs Markov partitions for an general surface diffeomorphism, the second one describes invariant measures for the horocyclic flow on general infinite hyperbolic surfaces. They both are typical of Omri Sarig’s work: the sets of hypotheses are natural and possible extensions could only be reached by radically new methods.

1. POSITIVE ENTROPY Diffeomorphisms on Surfaces

We consider a smooth closed surface $M$ and $f$ a $C^{1+\alpha}$-diffeomorphism of $M$. Recall the definition of the topological entropy of a continuous self-mapping $f$ of a metric compact space $(X,d)$. A subset $E \subset X$ is said to be $(n,\varepsilon)$-separated if, for $x \neq x' \in E$ and $i,0 \leq i < n$,

$$d(f^i x, f^i x') \geq \varepsilon.$$

The topological entropy $h_{\text{top}}$ is given by

$$h_{\text{top}} = h_{\text{top}}(X,f) = \lim_{\varepsilon \to 0} \limsup_{n} \frac{1}{n} \ln N(n,\varepsilon),$$

where $N(n,\varepsilon)$ is the maximal possible cardinality of a $(n,\varepsilon)$-separated set. In this section, we always assume that $h_{\text{top}}(M,f)$ is positive.

1.1. Results and background.

THEOREM 1.1. [31] Assume that there exists a measure with maximal entropy. Then there is an integer $p$ such that

$$\liminf_{k \to \infty} e^{-kp_{\text{top}}} p_{kp}(f) > 0,$$

where, for $n \in \mathbb{N}$, $P_n(f)$ is the number of hyperbolic periodic points of period $n$.

REMARKS. Theorem 1.1 might be wrong for zero entropy diffeomorphisms, for instance for an irrational translation on the torus. The hypothesis about the measure of maximal entropy is automatically satisfied if $f$ is a $C^\infty$ diffeomorphism of $M$ ([20]). The rate of growth of hyperbolic periodic points is a classical question, with several previously known properties. In particular:

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In two and higher dimensions, if \( f \) satisfies Axiom A, then (\([6]\)), on each basic set with maximal entropy, there is an integer \( p \) such that
\[
\lim_{k \to \infty} e^{-kph_{\text{top}}(f)} \cdot P_k(f) = 1
\]
and, outside of Axiom A, for a generic \( f \) in \( C^r, r \geq 2 \), with homoclinic tangencies, \( P_n(f) \) might grow arbitrarily fast (\([17]\)).

In dimension two, exponential growth of \( P_n \) has been proven by A. Katok (\([16]\)) for a general \( C^{1+a} \)-diffeomorphism of \( M \):
\[
\limsup_{n \to \infty} \frac{1}{n} \ln P_n(f) \geq h_{\text{top}}.
\]

Theorem 1.1 refines this last result and answers a question from \([16]\).

**Theorem 1.2** (\([30, 31]\)). There is at most a countable number of distinct ergodic measures of maximal entropy. For each one of them, say \( m \), the measured dynamical system \( (X, f; m) \) is measurably isomorphic to the product of a Bernoulli shift and a finite rotation.

**Remarks.** The first part of Theorem 1.2 is wrong for some zero entropy diffeomorphisms, for instance for the identity. Theorem 1.2 also holds for equilibrium states of Hölder-continuous functions, as soon as they have positive entropy (see \([30]\)). The same statement as in Theorem 1.2 but concerning the Lebesgue measure is a famous result of Pesin (\([24]\), see \([18]\) for the case of SRB measures).

Systems with a unique measure of maximal entropy were first considered by W. Parry (for transitive Markov shifts on a finite alphabet \([21]\)), K. Berg (for transitive automorphisms of compact groups \([4]\)) and B. Weiss (see \([36]\) for an early survey).

Theorem 1.2 answers a question from \([10]\) which was motivated by several known results:

- in all dimensions, for Axiom A systems \([6]\) there is at most a finite number of measures of maximal entropy,
- in dimension one, for piecewise monotonic mappings \([15]\) and for \( C^\infty \) mappings \([10]\), there is at most a finite number of ergodic measures of maximal entropy,
- the principal coding constructed by Boyle, Fiebig and Fiebig \([7]\) implies Theorem 1.2 for \( C^\infty \) mappings of compact surfaces.
- In dimension two, systems which can be coded by a Young tower (see \([37]\) and related papers) also have at most a countable number of ergodic measures of maximal entropy (see \([4]\) for the uniqueness for a family of Hénon mappings).

For these examples, the Bernoulli property follows from the weak Bernoulli property of the natural generating partition \([13]\).

A fundamental tool for proving these results is a symbolic representation of the dynamics by a finite or countable Markov shift. We recall the definition of a Markov shift. Let \( A \) be a countable alphabet, \( L \) a list of allowed two-letter words
in $A \times A$. The Markov shift $\Sigma$ is the subset of the set $A^\mathbb{Z}$ made of those infinite words with the property that all their two-letter subwords are in $L$. The set $\Sigma$ is endowed with the induced topology from the product topology on $A^\mathbb{Z}$. We shall assume that each letter can only enter in a finite number of words in $L$; then, the space $\Sigma$ is locally compact. We denote by $\Sigma^g$ the subset of $\Sigma$ made of words $\{x_n\}_{n \in \mathbb{Z}}$ such that the sequences $\{x_n\}_{n \geq 0}$ and $\{x_n\}_{n \leq 0}$ both contain at least a constant subsequence. Let $\sigma$ be the shift transformation on $\Sigma$. By the Poincaré Recurrence Theorem, for any $\sigma$-invariant probability measure $m$ on $\Sigma$, we have $m(\Sigma^g) = 1$. By a symbolic representation of high entropy measures, we mean the following:

**Theorem 1.3** (31). Let $0 < \chi < h_{top}$. There exist a locally compact Markov shift $\Sigma$ and a mapping $\varphi : \Sigma^g \to M$ such that

1. $\varphi \circ \sigma = f \circ \varphi$,
2. the mapping $\varphi$ is finite-to-one (but not bounded-to-one in general), Hölder-continuous and
3. the set $\varphi(\Sigma^g)$ has full measure for any invariant probability measure with entropy greater than $\chi$.

Moreover, there exists a mapping $\rho : A \times A \to \mathbb{N}$ such that, if $x = \{x_n\}_{n \in \mathbb{Z}} \in \Sigma^g$ with $(u, v) \in A \times A$ such that $x_n = u$ infinitely often (i.o.) for $n > 0$, and $x_n = v$ i.o. for $n < 0$, then the cardinality of $\rho^{-1}(\varphi x)$ is at most $\rho(u, v)$.

Theorems 1.1 and 1.2 follow from Theorem 1.3 by thermodynamic formalism. Indeed, any $f$ invariant ergodic measure on $M$ lifts to at most a finite number of ergodic $\sigma$-invariant measures on $\Sigma^g$ and those measures have the same entropy. On a transitive Markov shift admitting a measure of maximal entropy, Theorems 1.1 (with a limit 1) and 1.2 are obtained by thermodynamic formalism (the possible finite loss in the number of periodic points comes from the finite-to-one property). The coding Theorem 1.3 is a direct consequence of the construction of a Markov partition. The main technical result of (31) is:

**Theorem 1.4** (31). Let $M$ be a closed surface, $f$ a $C^{1+a}$ diffeomorphism of $M$ with positive entropy $h_{top}$ and $\chi, 0 < \chi < h_{top}$. Then, there is a collection $\mathcal{R} = \{R_i\}_{i \in \mathbb{N}}$ of pairwise disjoint subsets of $M$ with the following properties:

1. For any $f$ invariant ergodic measure $m$ with $h_m(f) > \chi$, $m(\cup_j R_j) = 1$.
2. Each $R_i$ has a local product structure: for $x \in R_i$ there are defined sets $W^s_i(x)$ (resp. $W^u_i(x)$) made of points $y \in R_i$ with $d(f^i x, f^i y) \to 0$ (resp. $d(f^{-i} x, f^{-i} y) \to 0$) as $i \to \infty$ such that all $x, y \in R_i, W^s_i(x) \cap W^u_i(y)$ exists and is a point of $R_i$.
3. Markov Property: if $x \in R_i$ and $f(x) \in R_j$, then $f(W^s_i(x)) \subset W^s_j(f(x))$ and if $x \in R_i$ and $f^{-i}(x) \in R_j$, then $f^{-i}(W^u_i(x)) \subset W^u_j(f^{-i}(x))$.

Given a collection $\mathcal{R} = \{R_i\}_{i \in \mathbb{N}}$ of pairwise disjoint sets as in Theorem 1.4, we define the subset $L$ of $\mathbb{N} \times \mathbb{N}$ by $(i, j) \in L$ if, and only if, $f(R_i) \cap R_j \neq \emptyset$. Then there is a mapping $\pi : \Sigma \to M$, where $\Sigma$ is the Markov shift defined by $\mathbb{N}$ and $L$. 

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defined by
\[ \pi(\{i_n\}_{n \in \mathbb{Z}}) = \bigcap_{n \in \mathbb{Z}} f^{-n} R_{i_n}. \]
Symbolic representations by compact Markov shifts appear in the works of R. Adler and B. Weiss [1] and of Y. Sinai [35]. Thermodynamic formalism for compact Markov shifts and its application to dynamics were developed by Y. Sinai, D. Ruelle, R. Bowen and others; this was extended to the infinite case by B. Gurevich, J. Aaronson, M. Denker, M. Urbański, R. D. Mauldin and O. Sarig, see the companion paper [25] and a recent detailed survey [32]. Observe that for \( C' \), \( r > 1 \) diffeomorphisms of surfaces, symbolic extension with a compact Markov shift can always be constructed [10] but that generically in the presence of homoclinic tangencies, the extension cannot be finite-to-one since it increases the entropy (see [12]).

A more precise version of Theorem 1.4 ensures that \( \Sigma \) is locally compact and that \( \pi \) has the properties of Theorem 1.3. We sketch the proof of Theorem 1.4 in the next subsection.

1.2. MARKOV PARTITION. Let us first recall the construction by Bowen of a Markov Partition in the Anosov case that works in any dimension (see e.g., [34, Chapter XI]). A \( C^{1+a} \)-diffeomorphism of \( M \) is said to be Anosov if there is an invariant decomposition \( TM = E^u \oplus E^s \) of the tangent bundle into two nontrivial subbundles and constants \( C > 0 \) and \( \lambda < 1 \) such that, for all \( n \geq 0 \),

\[
\text{for } v \in E^s, \| D f^n v \| \leq C \lambda^n \| v \| \quad \text{and for } v \in E^u, \| D f^{-n} v \| \leq C \lambda^n \| v \|.
\]

We need the following definitions:

- Given \( \varepsilon > 0 \), a sequence of points \( \{x_p, p \in \mathbb{Z}\} \) is an \( \varepsilon \)-pseudoorbit if for all \( p \), \( d(f(x_p), x_{p+1}) < \varepsilon \).
- Given \( \delta > 0 \), a sequence of points \( \{x_p, p \in \mathbb{Z}\} \) \( \delta \)-shadows \( y \in M \) if for all \( p \),
  \[ d(x_p, f^p(y)) < \delta. \]

Clearly, if two pseudoorbits \( \{x_p, p \in \mathbb{Z}\} \) and \( \{x'_p, p \in \mathbb{Z}\} \) \( \delta \)-shadow the same point \( y \), then \( d(x_p, x'_p) < 2\delta \) for all \( p \). The following properties were shown by Anosov (see [6, 34]):

1. For every \( \delta > 0 \), there is \( \varepsilon > 0 \) such that every \( \varepsilon \)-pseudoorbit \( \delta \)-shadows some point \( y \in M \).
2. For every \( \delta > 0 \), there is a finite set of points such that any point in \( M \) can be \( \delta \)-shadowed by some \( \varepsilon \)-pseudoorbit made with those points.
3. If \( \delta > 0 \) is small enough and \( \{f^p(y'), p \in \mathbb{Z}\} \) \( 2\delta \)-shadows \( y \), then \( y' = y \).

Property (3) decomposes into a ‘past’ and a ‘future’ shadowing: let \( W^s_{loc}(y) \) (resp. \( W^u_{loc}(y) \)) be the set of points \( y' \in M \) such that \( d(f^i y', f^i y) < 2\delta \) (resp. \( d(f^{-i} y', f^{-i} y) < 2\delta \)) for all \( i \geq 0 \). Then, if \( \delta \) is small enough, \( W^s_{loc}(y) \) and \( W^u_{loc}(y) \) are open smooth submanifolds, tangent respectively to \( E^s \) and \( E^u \).

Now, choose \( \delta > 0, \varepsilon > 0 \) so that all these properties are satisfied and a finite set of points \( \{x_i, i \in A\} \) satisfying property (2). Define the set \( L \) of allowed two-letter words by \( L := \{(i, j) \in A \times A : d(f(x_i), x_j) < \varepsilon\} \). Let \( \Sigma \) be the Markov shift.
defined by $A$ and $L$. By properties (1) and (3), there is a well-defined mapping
\( \pi : \Sigma \to M \) such that \( \pi([a_p]) \) is the only point in $M$ \( \delta \)-shadowed by the \( \varepsilon \)-pseudo-orbit \( \{x_n\}_n \).

For $\omega = \{a_p, p \in \mathbb{Z}\}$ a point of $\Sigma$, set
\[
[\omega]_0 = \{(a'_p)_{p \in \mathbb{Z}} : a'_0 = a_0\},
\]
\[
[\omega]_+ = \{(a'_p)_{p \in \mathbb{Z}} : a'_i = a_i \text{ for } i \geq 0\}
\]
and
\[
[\omega]_- = \{(a'_p)_{p \in \mathbb{Z}} : a'_j = a_j \text{ for } j \leq 0\}.
\]

Then, the family $\{\pi([\omega]_0), \omega \in \Sigma\}$ satisfies the properties (1), (2) and (3) of Theorem 1.4 with $W^s(\pi(\omega)) = \pi([\omega]_+)$, $W^u(\pi(\omega)) = \pi([\omega]_-)$. Moreover, $W^s(\pi(\omega)) \subset W^s_{loc}(\pi(\omega))$ and $W^u(\pi(\omega)) \subset W^u_{loc}(\pi(\omega))$. There is no guarantee that the sets $\{\pi([\omega]_0), \omega \in \Sigma\}$ are disjoint. A clever construction due to Sinai and Bowen ensures that one can subdivide the sets $\{\pi([\omega]_0), \omega \in \Sigma\}$ in a ‘Markovian’ way to obtain a covering $\mathcal{R} = \{R_i\}_{i \in \mathbb{N}}$ with properties (1), (2) and (3) of Theorem 1.4 and with the interiors of the $R_i$ pairwise disjoint.

We try to generalize this construction in the surface case. Consider, as in Theorem 1.4, a $C^{1+\alpha}$ diffeomorphism of a compact surface with positive entropy $h_{top}$ and \( \chi \) be a positive number smaller than $h_{top}$. By Ruelle inequality, any $f$-invariant ergodic probability measure $m$ with entropy greater than $\chi$ has two distinct Lyapunov exponents, one bigger than $\chi$, the other one smaller than $-\chi$. By Oseledec’s Theorem, the set of regular points $x \in M$ has full $m$ measure, where a point $x \in M$ is said to be regular if there is a decomposition of $T_x M = E^u_x \oplus E^s_x$ such that:

1. For $v \neq 0 \in E^u_x$, $\lambda_+ = \lim_{n \to +\infty} \frac{1}{n} \ln \|D_x f^n v\|$ exists and $\lambda_+ > \chi$,
2. For $v \neq 0 \in E^s_x$, $\lambda_- = \lim_{n \to -\infty} \frac{1}{n} \ln \|D_x f^n v\|$ exists and $\lambda_- < -\chi$,
3. $\lim_{n \to +\infty} \frac{1}{n} \ln |\det D_x f^n| = \lambda_+ + \lambda_-.$

Let $M_x$ be the set of such regular points. By definition the set $M_x$ is $f$-invariant, the decomposition $T_x M = E^u_x \oplus E^s_x$ is measurable and equivariant under $D_x f$. More is true: by Pesin theory, given $\varepsilon$ small, we can define a function $\ell$ on $M_x$, $\ell \leq 1$, such that:

1. $\ell(f^{\pm 1} x) \leq e^{\varepsilon} \ell(x)$,
2. if $W^u_{loc}(x)$ (resp. $W^s_{loc}(x)$) is the set of points $x' \in M$ such that $d(f^i x', f^i x) < 2 \varepsilon \ell(f^i x)$ (resp. $d(f^{-i} x', f^{-i} x) < 2 \varepsilon \ell(f^{-i} x)$) for all $i \geq 0$, then, $W^u_{loc}(x)$ and $W^s_{loc}(x)$ are open curves, tangent respectively to $E^u_x$ and $E^s_x$.
3. the angle between $E^u_x$ and $E^s_x$, the suitable H"older norm of the curves $exp^{-1}_{x} W^u_{loc}(x)$ and $exp^{-1}_{x} W^s_{loc}(x)$ in $(E^u_x, E^s_x)$ coordinates are all controlled by a power of $\ell(x)$.

The form of Pesin theory which is used here is recalled in the appendix of [31]. The global control of the H"older norm of the local invariant manifolds is new. We need a more general notion of local stable (resp. unstable) curves, which are exponential of $(E^u_x, E^s_x)$ graphs (resp. $(E^s_x, E^u_x)$ graphs) with size of domains and
The coding we have obtained is infinite-to-one, a priori, because a same point \( M \) to-one, namely, we want that a given point in \( M \) can be in many rectangles of different sizes. We want the coding to be finite-to-one, namely, we want that a given point in \( M \) can be coded by sequences \( \{x_i, i \in \mathbb{Z}\} \) with only a finite number of possible values of \( x_0 \). In some sense,
the gist of the Markov property is this finite-to-one property. Omri Sarig is led
to extend the definition of pseudoorbit by attaching to points on the manifold
symbols which describe more precisely the Pesin charts around the point. Then
a generalized pseudoorbit is a sequence of points and symbols such that the
points form an $\varepsilon$-orbit as above and that consecutive symbols satisfy an overlap-
ning comparison condition. A careful book-keeping of all the estimates allow
Omri Sarig to unravel the overdetermination of the direct naive coding and to
obtain a locally finite cover. The last step of the proof is a clever extension of
the final construction of Bowen and Sinai. The whole proof is a tour-de-force
and this presentation is an oversimplification of it.

2. Horocycle flow on infinite surfaces

In this section we consider a hyperbolic surface $M$. The unit tangent bundle
$SM$ can be identified with $\Gamma \sim PSL(2, \mathbb{R})$, where $\Gamma$ is a discrete torsion-free sub-
group of $PSL(2, \mathbb{R})$. In this identification, the horocycle flow $h_t, t \in \mathbb{R}$ is the ac-
tion of the one parameter group $h_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ by right translations. We want to de-
scribe all locally finite invariant measures under the horocycle flow. Clearly, the
Liouville measure, which becomes in the identification the trace on $\Gamma \sim PSL(2, \mathbb{R})$ of the Haar measure of $PSL(2, \mathbb{R})$, is invariant under all right translations, and
in particular by the horocycle flow. Also Lebesgue measures on closed orbits
of the horocycle flow, compact or not, are invariant as well. We call here trivial
invariant measures the combinations of measures proportional to the Liouville
measure or to Lebesgue measure on closed orbits of the horocycle flow. The
following are famous classical results:

- If $\Gamma$ is a uniform lattice, i.e, if $M$ is compact, then the Liouville measure is
  the only locally finite invariant measure (Furstenberg [14]);
- If $\Gamma$ is a lattice, i.e., $M$ has finite volume, then all locally finite invariant
  measures are trivial (Dani–Smillie [11]);
- the same is true for geometrically finite $\Gamma$ (Burger [8], Roblin [27] and
  Schapira [33]).

For a general $\Gamma$ with $M$ infinite, a particular case of Ratner’s results [26] is
that all finite invariant measures are trivial. But it turns out that there might be
other infinite, locally finite, invariant measures. The first examples were given
by Babillot and Ledrappier [5] in the case when $M$ is an abelian infinite cover of
a compact manifold $M_0$. The first result of Sarig’s on this topic is that, for infinite
abelian covers of compact manifolds, the measures exhibited in [5] are the only
ergodic locally finite invariant measures [28]. For more general manifolds, there
is the following definition:

**Definition 2.1.** A hyperbolic surface $M$ is said to have Babillot property if every
ergodic horocycle flow invariant locally finite measure that is not proportional to
the Lebesgue measure of a closed horocycle is quasi-invariant under the geodesic
flow.
The definition comes from the following observation by M. Babillot [2]: assume that an ergodic horocycle flow invariant locally finite measure $m$ is quasi-invariant under the geodesic flow and set $\tilde{m}$ for the lifted measure on $\text{PSL}(2, \mathbb{R})$. Using $KAN$ coordinates on $\text{PSL}(2, \mathbb{R})$, the measure $\tilde{m}$ can be written as

$$\tilde{m} = \nu \times e^{\alpha s} ds \times dt,$$

where $\alpha$ comes from the quasi-invariance under the geodesic flow and $\nu$ is a measure on $K$, the compact group of rotations. In the identification of $\text{PSL}(2, \mathbb{R})$ with the unit tangent bundle $S\mathbb{H}^2$ to the hyperbolic space, the measure $\nu$ is identified with a measure $\overline{\nu}$ on the circle at infinity $\partial \mathbb{H}^2$. For $\xi \in \partial \mathbb{H}^2$, let $P(z, \xi)$ be the Poisson kernel on $\mathbb{H}^2$ with pole at $\xi$. Set

$$F(z) = \int_{\partial \mathbb{H}^2} (P(z, \xi))^\alpha d\overline{\nu}(\xi). \quad (2.1)$$

Then, $F$ is a $\Gamma$-invariant positive $\alpha(\alpha - 1)$ eigenfunction of the Laplacian on $\mathbb{H}^2$. Moreover, if the manifold $M$ has the Babillot property, the formula (2.1) gives a one-to-one correspondence between nontrivial ergodic horocycle flow invariant locally finite measures $m$ and nontrivial minimal positive $\alpha(\alpha - 1)$ eigenfunctions of the Laplacian on $M$. Here, minimal means in the sense of the cone of positive $\alpha(\alpha - 1)$ harmonic functions: if a positive $\alpha(\alpha - 1)$ harmonic function is dominated by $F$, then it is proportional to $F$; trivial harmonic functions are the ones defined by a positive Eisenstein series $F(z) = \sum_n (P(z, \xi_n))^\alpha$, where $\xi_n$ is an orbit of $\Gamma$ made of parabolic points. In this language, the result of [28] is that abelian covers of compact surfaces have the Babillot property. This was extended to general covers of finite volume manifolds in [19].

In general, an infinite hyperbolic surface can be decomposed (in several ways) as a union of pairs of pants, or spheres with three punctures, with constant curvature -1 inside, with disjoint interiors and boundaries made of points or of simple closed geodesics called cuffs. The combinatorics of the decomposition and the lengths of the associated cuffs (a cusp is a cuff of length 0) form a part of the description of the manifold $M$. A surface is said to be tame if there exists a pants decomposition such that the lengths of the associated cuffs are uniformly bounded from above.

**Definition 2.2.** A hyperbolic surface is said to be weakly tame if there exists a pants decomposition such that for any geodesic ray $\gamma$ that crosses an infinite number of distinct cuffs $\{c_n(\gamma), n \geq 1\}$ associated with this decomposition, we have:

$$\liminf_{n} (\text{length } c_n(\gamma)) < \infty.$$

**Theorem 2.3 ([29]).** A weakly tame hyperbolic surface has the Babillot property.

Let $\delta(M)$ be the topological entropy of the geodesic flow on $SM$. It follows from the proof that tame hyperbolic surfaces satisfy $\delta(M) \geq 1/2$. Conversely:

**Theorem 2.4 ([29]).** Let $M$ be a hyperbolic manifold such that $\delta(M) < 1/2$ and such that every point in $\partial \mathbb{H}^2$ is accumulated by $\Gamma z_0$ for some (hence any) $z_0 \in \mathbb{H}^2$. Then $M$ does not have the Babillot property.
Manifolds satisfying the hypotheses of Theorem 2.4 have been constructed (see [23]). Moreover, in that case, for Lebesgue almost every $\xi \in \partial H^2$, we have

$$b(\xi) := \sum_{\gamma \in \Gamma} |\gamma'(\xi)| < \infty.$$ 

It follows that the ergodic components of the Liouville measure on $SM$ under the horocycle flow are exchanged, but not preserved, by the geodesic flow. Indeed, the measures $\mu_\sigma, \sigma \in \mathbb{R}$ defined in $KAN$ coordinates by

$$\int H(\xi, s, t) d\mu_\sigma := \int H(\xi, b(\xi) + \sigma, t) d\xi d\sigma$$

form a decomposition of the Haar measure on $PSL(2, \mathbb{R})$ into measures that are jointly invariant under the action of the horocycle flow and the left action of $\Gamma$. For different values of $\sigma$, the measures $\mu_\sigma$ are singular with respect to each other and are exchanged by the geodesic flow (up to a multiplicative constant).

The heart of the proof of Theorem 2.3 is to show that, if $M$ is weakly tame, such a situation cannot occur for any locally finite, ergodic, horocycle invariant measure $m$ because the equivalence classes of the relation $\mathcal{R}$ on $\partial H^2 \times \mathbb{R}$ are big in $m$ measure, where

$$((\xi, s), (\xi', s')) \in \mathcal{R} \iff \text{there is } \gamma \in \Gamma \text{ such that } \xi' = \gamma(\xi), s' = s - \ln|\gamma'(\xi)|.$$ 

The precise statement is the Holonomy Lemma 2.1.1 and most of [29] consists of the proof of this Holonomy Lemma.

REFERENCES


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