Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity. You must justify carefully any argument your use. Correct answers without supporting proof will be given no credit. You may use standard results without proof, provided that you state them clearly and completely (e.g. stating the name of a theorem is not sufficient). If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $V$ be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that there is a vector $v \in V$ such that the list
   \[
   \{v, Tv, T^2v, \ldots \}
   \]
   spans $V$. Show that if $S : V \to V$ is a linear transformation that commutes with $T$, then there is a polynomial $f$ such that $S = f(T)$.

2. Prove that there exist two non-similar invertible $3 \times 3$ real matrices $A$ which satisfy
   \[
   A^{-1} = A^2 + A,
   \]
   and give a representative of each similarity class.

3. A matrix is called stable if each of its eigenvalues has negative real part. Assume that $A$ is a $3 \times 3$ real matrix with negative entries on the main diagonal, i.e. $A_{ii} < 0$ for all $i = 1, 2, 3$. Prove that if $A$ is stable, then there exists a pair of indices, $i \neq j$ such that the $2 \times 2$ principal sub-matrix
   \[
   B^{(i,j)} = \begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{pmatrix},
   \]
   is also stable.

4. Let $A$ be a real $n \times n$ matrix, such that $A^T = A$. Assume that all the diagonal elements of $A$ satisfy $A_{ii} = n - 1$, while the off-diagonal elements satisfy $|A_{ij}| < 1$ for all $i \neq j$. Denote $B = A + A^{-1} - 2I$. Prove that the limit
   \[
   P = \lim_{t \to +\infty} \exp(-tB)
   \]
   exists and is equal to the orthogonal projector to the eigenspace $V_1 = \ker(A - I)$. 
