Qualifying Examination in Functional Analysis

January 11, 2021

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity. You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

Note: In the following problems, all spaces are over the field of real numbers.

1. Let $1 \leq p < \infty$. Consider the Banach space
   \[ \ell^p = \left\{ (x_n)_{n \geq 1} : x_n \in \mathbb{R} \text{ for all } n \text{ and } \|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty \right\} . \]
   Let $(a_n)_{n \geq 1}$ be a bounded sequence of real numbers and let $A$ be the bounded linear operator on $\ell^p$ given by $A((x_n)_{n \geq 1}) = (a_n x_n)_{n \geq 1}$. Prove that $A$ is compact if and only if $\lim_{n \to \infty} a_n = 0$.

2. Let $M$ be a metric space, let $X$ be a Banach space, and let $f$ be a function from $M$ to $X$. Prove that if $\varphi \circ f$ is Lipschitz continuous for every $\varphi \in X^*$, then $f$ is Lipschitz continuous.

3. Let $H$ be a Hilbert space, and let $f_1, f_2 \in H$ be two linearly independent vectors. Consider the closed, convex subset
   \[ S = \left\{ u \in H : (u, f_1) \leq 0, \quad (u, f_2) \leq 0 \right\} . \]
   For a given $f \in H$, let $g$ be the perpendicular projection of $f$ on the set $S$, so that
   \[ g \in S, \quad \|g - f\| = \min_{u \in S} \|u - f\| . \]
   Prove that
   \[ f - g = c_1 f_1 + c_2 f_2 , \quad \text{for some } c_1, c_2 \in \mathbb{R} . \]

4. Let $H$ be a Hilbert space. Prove that the following are equivalent:
   (i) $H$ is infinite dimensional.
   (ii) For every vector $y \in H$ with $\|y\| \leq 1$, there exists a sequence of unit vectors $(x_n)_{n \geq 1}$ which converges weakly to $y$. 