Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity. You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

In the problems below, $\mathbb{D}$ denotes the unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

1. Let $f$ be a holomorphic function from $\mathbb{D}$ into $\mathbb{D}$ such that $f(\alpha) = \beta$ and $f(\beta) = \alpha$ for some $\alpha \neq \beta$ in $\mathbb{D}$. Prove that $f \circ f$ is the identity map.

2. Prove that there is no sequence of entire functions $\{f_n\}$ that converges to $f(z) = z^{-3}$ uniformly on the set $A = \{z : \frac{1}{2} < |z| < 2\}$.

3. Let $U = \{z : \text{Im } z > 0 \text{ and } |z - i| > 1\}$.
   Find a conformal equivalence $f$ from $U$ onto the unit disc $\mathbb{D}$.
   You may represent $f$ as a composition of several maps.
   Give a formula and draw the domain and range for each of the maps.

4. Use contour integration to evaluate the integral $\int_0^\infty \frac{\sqrt{x} \log x}{1+x^2} \, dx$
   where $\log x$ is the natural logarithm of $x$. 