**Instructions:** To pass the exam you must correctly solve at least two of the following four problems.

Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $H$ be a subgroup of a group $G$ of index 3. Prove that either $H$ is normal, or that $H$ has a subgroup $N$ of index 2 in $H$ such that $N$ is normal in $G$.

2. Describe the maximal ideals $m$ of the polynomial ring $\mathbb{Z}[x]$ that contain the integer 30 and the polynomial $x^2 + 1$. Write down two explicit generators for each such ideal. How many such maximal ideals are there?

3. Let $R = \mathbb{Z}[i]$ denote the ring of Gaussian integers, i.e., $i^2 = -1$. Let $M \cong R^3$ be the free $R$-module of rank 3. Let $N \subseteq M$ be the submodule generated by $(1, i, 4)$ and $(2, 3 + 2i, 14)$. Express $M/N$ explicitly as a direct sum of cyclic $R$-modules.

4. Let $K$ be a field, and let $L$ be an extension field of $K$. Let $u \in L$, and assume that the minimal polynomial of $u$ over $K$ is $x^n - \alpha$ for some $\alpha \in K$. Let $n = md$ for positive integers $m, d$.

   1. Show that $[K(u^m) : K] = d$.

   2. What is the minimal polynomial of $u^m$ over $K$?