Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity. You must justify carefully any argument your use. Correct answers without supporting proof will be given no credit. You may use standard results without proof, provided that you state them clearly and completely (e.g. stating the name of a theorem is not sufficient). If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Determine the Galois group of the splitting field of the polynomial $x^6 + 3$ over $\mathbb{Q}$.
   **Hint:** Suppose that $\alpha$ is a root of $x^6 + 3$. Show that $\beta := (\alpha^3 + 1)/2$ is a primitive 6th root of unity.

2. Let $R$ be a UFD, and let $f$ be any non-zero, non-unit irreducible element of $R$. Prove that $R/(f)$ is also a UFD or disprove by counterexample. What happens if $R$ is also a PID?

3. A group $G$ is said to act faithfully on a set $X$ of cardinality $n$ if the corresponding homomorphism $\phi : G \to S_n$ is injective. Show that an abelian group $G$ of order 100 cannot act faithfully on a set with 13 elements.
   **Hint:** Consider the 5-Sylow subgroup $H$ of $G$ and determine its centralizer in $S_{13}$.

4. Let $R$ be a PID. Let us consider the quotient $M = R[t]/t^2R[t]$ as a module over the polynomial ring $R[t]$. Prove that $M$ is not isomorphic to a direct sum $M_1 \oplus M_2$ of two nonzero $R[t]$-modules $M_1$ and $M_2$. 