PhD Qualifying Examination

TOPOLOGY

December 9, 2018

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You must justify carefully any argument your use. Correct answers without supporting proof will be given no credit. You may use standard results without proof, provided that you state them clearly and completely (e.g. stating the name of a theorem is not sufficient). If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. (Tube Lemma) Let $X$ and $Y$ be topological spaces and suppose that $X$ is compact. Suppose that $y \in Y$ is such that $X \times \{y\} \subset U$ for some open set $U \subset X \times Y$. Prove that there exists an open set $W \subset Y$ such that $X \times \{y\} \subset X \times W \subset U$.

2. Prove that the fundamental group of a topological group is abelian.

3. Let $S^n$ denote the $n$-sphere and let $f: S^n \to S^n$ be a continuous map with degree $\deg f \neq (-1)^{n+1}$. Prove that $f$ has a fixed point.

4. Let $p$ be a point in the two-torus $T^2$. Compute the homology of $T^2 \setminus \{p\}$ using the Mayer–Vietoris exact sequence.