Instructions: To pass the exam you must correctly solve at least two of the following four problems.

Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

1. Let $A$ be a set in $\mathbb{R}$ of Lebesgue measure greater than 1. Prove that there exist $x, y \in A$ such that $x - y$ is a positive integer.

2. Let $(X, A, \mu)$ be a measure space with $\mu(X) = 1$. Suppose a family $f_n \in L^2(X, A, \mu)$ satisfies

$$1 \leq \|f_n\|_2 \leq \|f_n\|_1/\alpha$$

for some real number $\alpha > 0$. Prove that the set of points $x$ such that $|f_n(x)| \geq \alpha/2$ for infinitely many numbers $n$ has measure at least $\alpha^2/4$.

3. Let $f_n \in C[0,1]$. Assume that for every $x \in [0,1]$ there exists a finite limit $f(x) = \lim f_n(x)$. Apply the Baire category theorem to the sets

$$A_n = \{x : |f_m(x) - f_n(x)| \leq 1/3 \forall m > n\}$$

to prove that $f$ cannot be the salt and pepper function

$$g(x) = \begin{cases} 
1, & \text{if } x \in \mathbb{Q}, \\
0, & \text{otherwise.}
\end{cases}$$

4. Suppose $\nu \ll \mu$ on $(X, A)$ and $\mu(X) = \nu(X) = 1$. Show that if a sequence of measurable functions $f_n$ converges in measure $\mu$ to a function $f$, then it converges to $f$ in measure $\nu$ as well.