Instructions: To pass the exam you must correctly solve at least two of the following four problems. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly.

If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

In the problems below, \( \mathbb{D} \) denotes the unit disc \( \{ z \in \mathbb{C} : |z| < 1 \} \).

1. Suppose that a function \( f \) is holomorphic in \( \mathbb{D} \), continuous in \( \overline{\mathbb{D}} \), and satisfies \( |f(z)| = 1 \) for all \( z \) with \( |z| = 1 \). Prove that either \( f \) is constant or \( f(\mathbb{D}) = \mathbb{D} \).

2. Let \( a > 0 \). Use contour integration to show that
\[
\int_0^\infty \frac{\log x}{a^2 + x^2} \, dx = \frac{\pi}{2a} \log a.
\]

3. Let \( U \) be the region between the circles \( C_1 = \{ z : |z| = 1 \} \) and \( C_2 = \{ z : |z - \frac{1}{2}| = \frac{1}{2} \} \), that is, \( U = \{ z : |z| < 1 \text{ and } |z - \frac{1}{2}| > \frac{1}{2} \} \).
Find a conformal equivalence \( f \) from \( U \) onto the unit disc \( \mathbb{D} \).
You may represent \( f \) as a composition of several maps.
Give a formula and draw the domain and range for each of the maps.

4. Determine the change of the argument of the function
\[
f(z) = z^2 + 1 + \frac{5}{z} + \frac{1}{z^2}
\]
as \( z \) traverses the unit circle once counterclockwise.