PhD Qualifying Examination
ABSTRACT ALGEBRA
December 8, 2018

Instructions: To pass the exam you must correctly solve at least two of the following four problems. Only the two highest of your overall scores on the individual problems will be counted. So under most circumstances you should concentrate your effort on two solutions. Your solutions will be evaluated for correctness, completeness and clarity.

You may use standard results without proof, provided that you state them clearly. If you have any question about whether a particular result may be used without proof, please ask the faculty member proctoring the exam.

(1) Let $G$ and $H$ be finite groups. Show that every $p$-Sylow subgroup of $G \times H$ is of the form $P \times Q$ for $P$ a $p$-Sylow subgroup of $G$ and $Q$ a $p$-Sylow subgroup of $H$.

(2) Let $R$ be a principal ideal domain. Let $a$ and $b$ be elements of $R$. Prove that the $R$-module of $R$-module homomorphisms from $R/aR$ to $R/bR$ is 0 if $a \neq 0$ and $b = 0$. Prove that it is isomorphic to $R/\gcd(a,b)R$ when $a$ and $b$ are both nonzero. In each case, describe explicitly a homomorphism that generates this cyclic module by specifying its value on the image of 1 in $R/aR$.

(3) Let $\zeta = e^{\pi i/6}$.
   (a) Find the minimal (monic) polynomial of $\zeta$ over $\mathbb{Q}$.
   (b) Find the Galois group of $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
   (c) Write each subfield of $\mathbb{Q}(\zeta)$ in the form $\mathbb{Q}(u)$ (for an explicit $u$).

(4) Let us consider the ring
    \[ \mathcal{O} = \{a + 2b\sqrt{6} \mid a, b \in \mathbb{Z}\}. \]
    Prove that the multiplicative group $\mathcal{O}^*$ of invertible elements of $\mathcal{O}$ is infinite.