Additional Topics in Algebra

Objective 1: Absolute Values of Functions

Recall that the **absolute value function** $f(x) = |x|$ can be defined by a piecewise function:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Therefore, if $g(x)$ is a function, the composition $|g(x)|$ can be defined as:

$$|g(x)| = \begin{cases} g(x) \text{ if } g(x) \geq 0 \\ -g(x) \text{ if } g(x) < 0 \end{cases}$$

Example: Given the function $g(x) = x^2 + x - 2$:

a. For what values of x is $g(x) \geq 0$? For what values of x is $g(x) < 0$?

b. Express $|g(x)|$ as a piecewise function.

c. Find $|g(-3)|, |g(-1)|, |g(0)|$, and $|g(1)|$.

Objective 2: Graphing Absolute Values of Functions

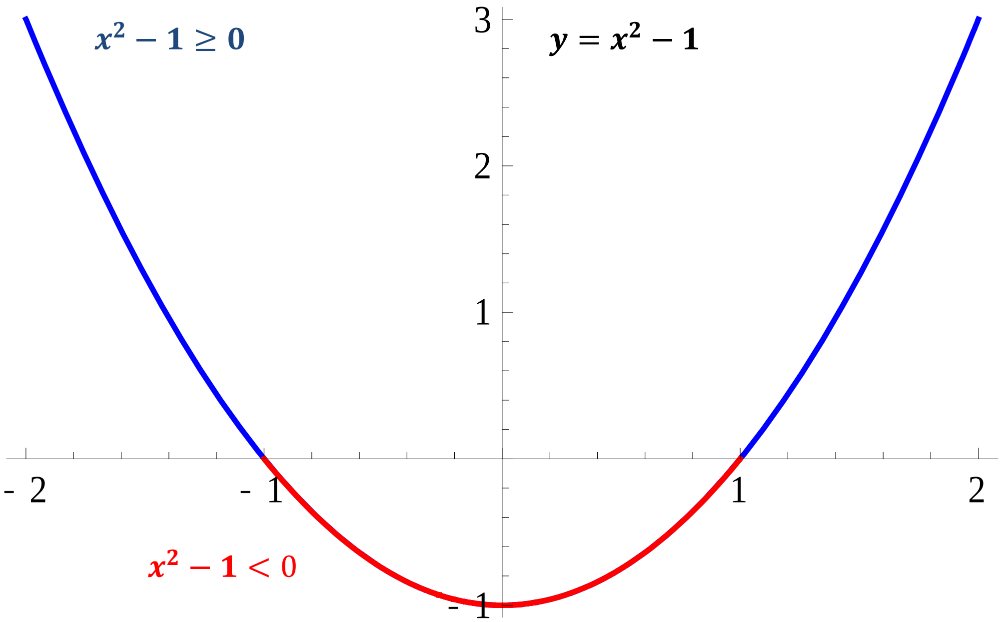
Using the piecewise definition

$$|g(x)| = \begin{cases} g(x) \text{ if } g(x) \geq 0 \\ -g(x) \text{ if } g(x) < 0 \end{cases}$$

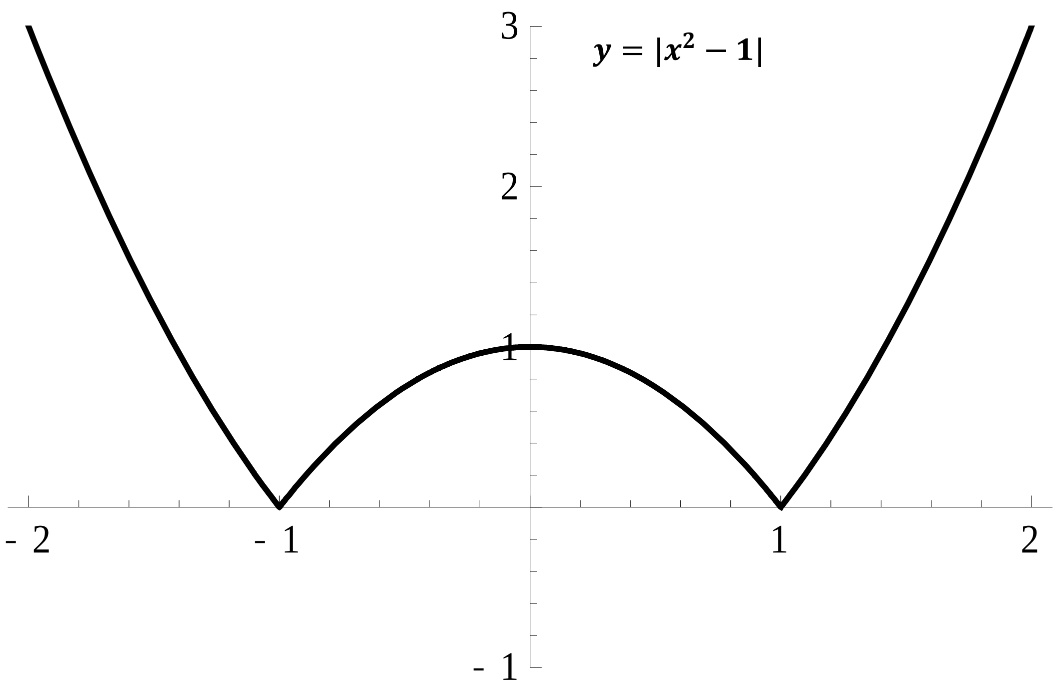
and knowledge of graph transformations, we can adopt a strategy for sketching the graph of the absolute value of a function.

Example: Sketch the graph of $y = |x^2 - 1|$.

Step 1. Sketch the graph of $g(x)$. Take note of where $g(x) < 0$ (the graph lies below the *x*-axis) and where $g(x) \geq 0$.



Step 2. Reflect across the *x*-axis every point on the graph of $g(x)$ that lies below the *x*-axis; that is to say, every point $(x, g(x))$ with $g(x) < 0$ should be reflected to $\left(x, -g(x)\right)$.



Examples: Sketch the graphs of the following functions:

1. $f(x) = |x^2 + x - 2|$

2. $g(x) = \left|\frac{1}{x}\right|$

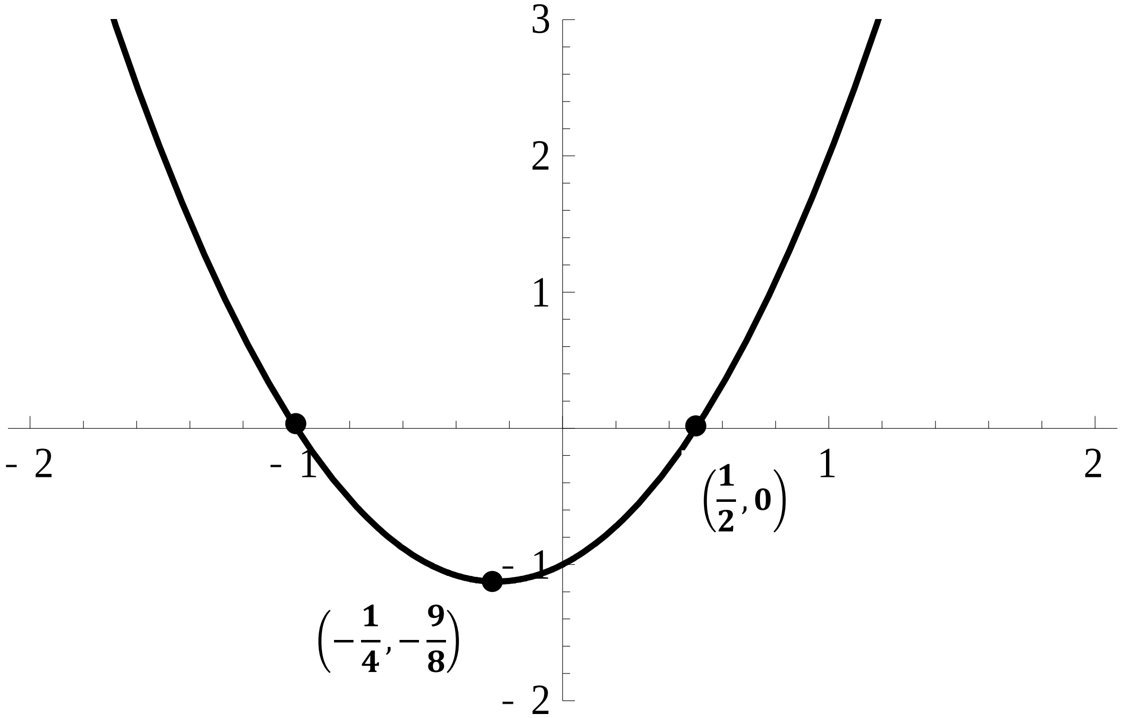
3. $h(x) = |-x^3 - 6x^2 - 9x|$

4. $k(t) = \left|\frac{t^2 - 4}{t - 1}\right|$

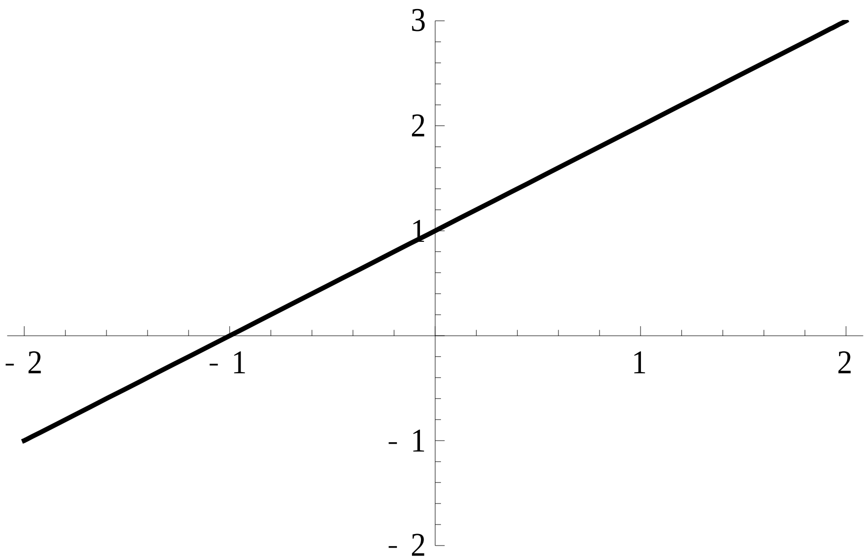
Objective 3: Graphing Multiple Functions on One Set of Axes

Example: Draw the graphs of $y = 2x^2 + x - 1$ and $y = x + 1$ on the same set of axes.

First, graph $y = 2x^2 + x - 1 = (2x - 1)(x + 1)$. The graph is a parabola, opening upward, with vertex $(-{}^1/_4, -{}^9/_8)$ which crosses the *x*-axis at $x = -1$ and $x = {}^1/_2$.



The graph of $y = x + 1$ is a line with slope 1 and *y*-intercept (0,1).



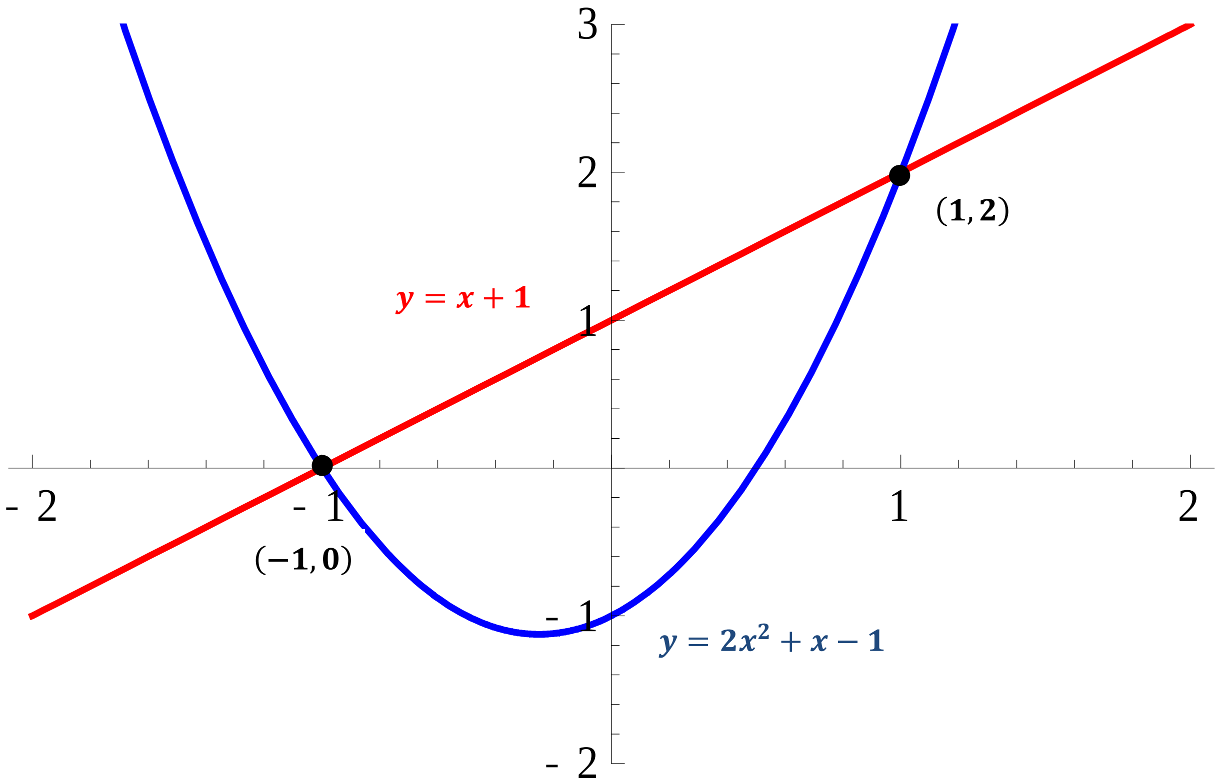
In order to accurately draw both graphs on one set of axes, we will need to know where they **intersect**.

Definition. The point (a, b) is an **intersection point** for the graphs of $y = f(x)$ and $y = g(x)$ if a is in the domains of f and g and $f(a) = g(a)$.

To find intersection points, we set the functions equal to each other and solve for x.

$$2x^2 + x - 1 = x + 1$$
$$2x^2 - 2 = 0$$
$$2(x + 1)(x - 1) = 0$$
$$x = -1, x = 1$$

The intersection points are $(-1,0)$ and $(1,2)$.



Examples. Graph each pair of functions on one set of axes.

1. $y = \sqrt{x}$ and $y = x - 6$

2. $y = x^3 + 2x^2 + x + 2$ and $y = 2$

3. $y = -x^3$ and $y = -x$

4. $y = \dfrac{1}{x+1}$ and $y = -2x + 1$