

1.6.10. Find all solutions to $x^6 - 7x^3 = 8$.

Rewrite as $x^6 - 7x^3 - 8 = 0$ and let $u = x^3$. Equation becomes

$$u^2 - 7u - 8 = 0$$

$$(u - 8)(u + 1) = 0$$

so $u = 8$ or $u = -1$. Solving for x :

$u = 8$ means $x^3 = 8$, or $x^3 - 8 = 0$. Factoring the difference of cubes:

$$(x - 2)(x^2 + 2x + 4) = 0.$$

$x - 2 = 0$ when $x = 2$, and $x^2 + 2x + 4 = 0$ when $x = -1 \pm i\sqrt{3}$ (quadratic formula).

Similarly, $u = -1$ means $x^3 + 1 = 0$, or

$$(x + 1)(x^2 - x + 1) = 0 \text{ (sum of cubes factorization)}$$

and $x + 1 = 0$ when $x = -1$, $x^2 - x + 1 = 0$ when $x = \frac{1 \pm i\sqrt{3}}{2}$ (quadratic formula)

Therefore, solution set is $\left\{-1, 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}\right\}$.