A Very Brief Introduction to Conservation Laws

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A conservation law is an PDE that describes time evolution of some quantity (quantities) that is (are) conserved in time.

Let \( u(t, x) \) be the unknown. We have the initial value problem

\[
\frac{\partial}{\partial t} u(t, x) + \frac{\partial}{\partial x} f(u(t, x)) = 0, \quad u(0, x) = \bar{u}(x).
\]

If \( u(t, x) \) is a scalar function, then \( f(\cdot) \) is a scalar-valued function. Then, the equation is called a scalar conservation law.

If \( u(t, x) \in \mathbb{R}^n \) is a vector of length \( n \), the \( f(\cdot) \) is a vector-valued function. Then, this is a system of conservation laws.

The function \( f(u) \) is called the flux. Typically it is a non-linear function.
A conservation law describing traffic flow

\[ \rho = \text{density of cars} \]

\[ \int_a^b \rho(t, x) \, dx = \text{total number of cars at time } t \text{ within the interval } [a, b] \]

\[ \frac{d}{dt} \int_a^b \rho(t, x) \, dx = [\text{flux of cars entering at } a] - [\text{flux of cars exiting at } b] \]

\[ = f(t, a) - f(t, b) \]
**flux:**  
= [number of cars crossing the point $x$ per unit time] 

= [density] × [velocity] 

= $\rho(t, x) \cdot v(t, x)$ 

Assume: $v = v(\rho)$, i.e., speed depends on the density $\rho$. 

For example: $v(\rho) = k(M - \rho), \quad 0 \leq \rho \leq M$ 

$M$=max car density 

$kM$=max car speed when $\rho = 0$. 

For simplicity, we choose $k = 1$, $M = 1$ and get $v(\rho) = 1 - \rho$. 

Flux: $f(\rho) = \rho \cdot v(\rho) = \rho(1 - \rho)$. 

$$\frac{d}{dt} \int_{a}^{b} \rho(t, x) \, dx = f(\rho(t, a)) - f(\rho(t, b)).$$ 

This is an integral form of the conservation law.
\[
\frac{d}{dt} \int_a^b \rho(t, x) \, dx = f(\rho(t, a)) - f(\rho(t, b))
\]

Integrate in time from \(t_1\) to \(t_2\):

\[
\int_a^b \rho(t_2, x) \, dx - \int_a^b \rho(t_1, x) \, dx = \int_{t_1}^{t_2} f(\rho(t, a)) \, dt - \int_{t_1}^{t_2} f(\rho(t, b)) \, dt,
\]

or equivalently

\[
\int_a^b \rho(t_2, x) \, dx = \int_a^b \rho(t_1, x) \, dx + \int_{t_1}^{t_2} f(\rho(t, a)) \, dt - \int_{t_1}^{t_2} f(\rho(t, b)) \, dt.
\]

This gives an expression for the mass in \([a, b]\) at \(t_2\) in terms of the mass at an earlier time \(t_1\) and the total integrated flux along the boundary.
Differential form of the conservation law

Integral form

\[\int_a^b \left[ \rho(t_2, x) - \rho(t_1, x) \right] dt = \int_{t_1}^{t_2} \left[ f(\rho(t, a)) - f(\rho(t, b)) \right] dt,\]

Assume that \(\rho(x, t)\) is a differentiable function in both \(x\) and \(t\), so

\[\rho(t_2, x) - \rho(t_1, x) = \int_{t_1}^{t_2} \frac{\partial}{\partial t} \rho(t, x) dt\]

\[f(\rho(t, b)) - f(\rho(t, a)) = \int_a^b \frac{\partial}{\partial x} f(\rho(t, x)) dx\]

Then

\[\int_{t_1}^{t_2} \int_a^b \left\{ \frac{\partial}{\partial t} \rho(t, x) + \frac{\partial}{\partial x} f(\rho(t, x)) \right\} dx \, dt = 0.\]
\[ \int_{t_1}^{t_2} \int_{a}^{b} \left\{ \frac{\partial}{\partial t} \rho(t, x) + \frac{\partial}{\partial x} f(\rho(t, x)) \right\} \, dx \, dt = 0. \]

This holds for all \(a, b\) and \(t_1, t_2\)! The integrand must be 0!

\[ \frac{\partial}{\partial t} \rho(t, x) + \frac{\partial}{\partial x} f(\rho(t, x)) = 0, \]

or with simplified notation

\[ \rho_t + f(\rho)_x = 0. \]

Traffic flow model:

\[ \rho_t + (\rho(1 - \rho))_x = 0, \quad \rho(0, x) = \rho_0(x). \]
System of conservation laws: gas dynamics

The most celebrated example for a system of conservation laws comes from gas dynamics, with the famous **Euler equations**.

Variables: \( \rho \) = density, \( v \) = velocity, \( \rho \cdot v \) = momentum, \( E \) = energy, \( p \) = the pressure.

Three conserved quantities: conservation of mass, momentum and energy.

These exactly give us the following \( 3 \times 3 \) system

\[
\begin{align*}
\rho_t + (\rho v)_x &= 0, & \text{conservation of mass} \\
(\rho v)_t + (\rho v^2 + p)_x &= 0, & \text{conservation of momentum} \\
E_t + (v(E + p))_x &= 0, & \text{conservation of energy.}
\end{align*}
\]

There is an additional equation, where the pressure \( p \) is given as a function of other quantities. This is called the “equation of the state”. For example, \( p = p(\rho, v) \).
Simplest Case: Linear Transport equation

Consider

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x). \]

If \( f(u) = au + b \) (linear function), then

\[ u_t + a \cdot u_x = 0, \quad u(0, x) = u_0(x). \]

This is the transport equation. Explicit solution

\[ u(t, x) = u_0(x - at). \]

One can easily verify this but plug it into the equation, and also check the initial condition.

The solution is simply the initial profile \( u_0(x) \) traveling with constant velocity \( a \).
Consider nonlinear flux $f \in C^2$ for

$$u_t + f(u)_x = 0, \quad \Rightarrow \quad u_t + f'(u)u_x = 0.$$ 

A characteristic is a line $t \mapsto x(t)$ such that $x'(t) = f'(u(t, x(t)))$.

The evolution of $u$ along a characteristic:

$$\frac{d}{dt} u(t, x(t)) = u_t + u_x x'(t) = -f'(u)u_x + u_x f'(u) = 0.$$ 

$\Rightarrow$ $u$ is constant along a characteristic!

$\Rightarrow$ $x'(t) =$constant along a characteristic!

$\Rightarrow$ all characteristics are straight lines, with slope=$f'(u(0, x))$!
Traffic Flow: Speed of cars and characteristic speed

\[ \rho_t + f(\rho)_x = 0, \quad f(\rho) = \rho \nu(\rho) = \rho(1 - \rho), \quad \rho(0, x) = \bar{\rho}(x) \]

\[ \nu(\rho) = 1 - \rho = \text{speed of cars, depending on the density} \]

characteristic speed \( = f'(\rho) = 1 - 2\rho \leq \nu(\rho) \)

characteristic speed is not the same as the car speed!

Characteristics \( t \mapsto x(t) \) are lines where information is carried along!

\[ \rho(t, x(t)) = \bar{\rho}(x(0)) = \text{constant} \]
Characteristics and particle trajectories

\[ \rho(t) \]

\[ p(t) \]

\[ \bar{\rho}(x) \]

\[ x(t) \]

\[ x_0 \]

\[ \rho(\tau,x) \]

\[ \tau \]

\[ t \]

\[ x \]
Loss of singularity for nonlinear equations

Points on the graph of $\rho(t, \cdot)$ move horizontally, with characteristic speed $f'(\rho)$. At a finite time $\tau$ the tangent becomes vertical and a discontinuity form.
Formation of Shock waves in finite time

For general scalar conservation law

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x) \]

if \( f \) is a nonlinear function, i.e., \( f'(u) \) is not constant, then, characteristics initiated at different point of \( x \) at \( t = 0 \) will have different slope, and they will interact in finite time.

→ discontinuities will form in finite time even with smooth initial data!

→ These are called shock waves or shocks!

→ We must only require \( u(t, x) \) bounded and measurable.
Discontinuous solutions will not satisfy the differential equation

\[ u_t + f(u)_x = 0 \]

Re-define solution concept, using integral form.

\[
\int_a^b \left[ u(t_2, x) - u(t_1, x) \right] dt = \int_{t_1}^{t_2} \left[ f(u(t, a)) - f(u(t, b)) \right] dt,
\]

\( u(t, x) \) is a **weak solution** if the integral form holds for any \( a, b, t_1, t_2 \).

An alternative definition: \( u = u(t, x) \) is a weak solution if

\[
\iint \left\{ u\phi_t + f(u)\phi_x \right\} dx \, dt = 0
\]

for every positive test function \( \phi \in C^1 \) with compact support.
Shock propagation; Riemann Problem

Riemann problem: \( u_t + f(u)_x = 0 \)
\[
\begin{align*}
    u(0, x) &= \begin{cases} 
        u^l, & x \leq 0 \\
        u^r, & x > 0
    \end{cases}
\end{align*}
\]

Shock speed: let \( s \) be the shock speed, and let \( M > st \).

\[
    u(t, x) = \begin{cases} 
        u^l, & x < st \\
        u^r, & x > st
    \end{cases}
\]

\[
    \int_{-M}^{M} u(x, t) \, dx = (M + st)u^l + (M - st)u^r
\]

\[
    \frac{d}{dt} \int_{-M}^{M} u(x, t) \, dx = su^l - su^r = s(u^l - u^r)
\]

(by conservation law:)

\[
    \frac{d}{dt} \int_{-M}^{M} u(x, t) \, dx = f(u^l) - f(u^r).
\]

Rankine-Hugoniot jump condition:

\[
    s(u^l - u^r) = f(u^l) - f(u^r)
\]
Manipulating conservation laws

Burgers’ equation:

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad f(u) = \frac{1}{2} u^2, \quad f'(u) = u, \quad f''(u) = 1 > 0 \quad (1) \]

Shock speed:

\[ s_1 = \frac{f(u^l) - f(u^r)}{u^l - u^r} = \frac{1}{2} \frac{(u^l)^2 - (u^r)^2}{u^l - u^r} = \frac{1}{2} (u^l + u^r). \]

Multiply Burgers equation by 2\(u\),

\[ 2u \cdot u_t + 2u \cdot uu_x = 0, \quad (u^2)_t + \left( \frac{2}{3} u^3 \right)_x = 0, \quad (2) \]

Shock speed

\[ s_2 = \frac{2}{3} \left( \frac{(u^l)^3 - (u^r)^3}{(u^l)^2 - (u^r)^2} \right), \quad s_1 \neq s_2. \]

(1) and (2) are equivalent for smooth solutions, but very different for discontinuous solutions!
The equal area rule

Method of characteristics leads to multi-valued functions, after finite time. To get back to single-valued functions, we inserting a shock. The exact location of the shock follows the “equal area rule”.

\[ \text{Diagram showing the equal area rule.} \]
**Observation**: Following the equal area rule, the characteristics enter the shock both from the left and from the right.

**Burgers’ equation:**

\[ x = st \]
Admissible conditions: Lax’s entropy condition

\( x(t) \): location of a shock

\( u_l = u(t, x-) \), \( u_r = u(t, x+) \): left and right state of the shock.

Lax’s entropy condition:

\[ f'(u_l) > x'(t) > f'(u_r). \]
Admissible shocks for Burgers equation

\[ f(u) = \frac{u^2}{2}, \quad f'(u) = u, \quad f''(u) = 1 > 0 \]

Lax’s condition \( f'(u_l) > f'(u_r) \) implies \( u_l > u_r \).

Conclusion: Only downward jumps are admissible.

In general, if \( f''(u) > 0 \) (convex), then \( f'(u) \) is increasing, then only downward jumps are admissible.
Admissible shocks for traffic flow

\[ f(\rho) = \rho(1 - \rho), \quad f'(\rho) = 1 - 2\rho, \quad f''(\rho) = -2 < 0. \]

Lax’s condition \( f'(\rho_l) > f'(\rho_r) \) implies \( \rho_l < \rho_r \). Only upward jumps are admissible.

In fact, image cars lined up in front a red light (discontinuous initial data with a downward jump). At \( t = 0 \), the light turn green. We never observe this jump in the car density moves forward. Actually, we observe that the cars spread out.

In general, for

\[ u_t + f(u)_x = 0, \]

if \( f''(u) < 0 \) (concave), then \( f'(u) \) is decreasing, so upward jumps are admissible.
Burgers’ equation:
**Rarefaction waves**

convex flux: \( u_t + f(u)_x = 0, \quad u(0, x) = \begin{cases} u^l, & x \leq 0, \\ u^r, & x > 0, \end{cases} \quad u^l < u^r. \)

Self similar solution: \( u \) depends only on \( \xi = x/t \). Rarefaction fan.

\[ u(t, x) = \begin{cases} u^l, & (x/t) < f'(u^l) \\ u^r, & (x/t) > f'(u^r) \\ \phi(x/t), & f'(u^l) \leq (x/t) \leq f'(u^r) \end{cases} \quad \phi(\xi) : \text{smooth function} \]
Case study: traffic jam

\[ \rho_t + f(\rho)_x = 0, \quad f(\rho) = \rho(1 - \rho) \]

Riemann problem: \[ \rho(0, x) = \begin{cases} 0.5, & x < 0 \\ 1, & x > 0 \end{cases} \]

Shock speed: \[ s = \frac{f(1) - f(0.5)}{1 - 0.5} = \frac{0 - 0.25}{0.5} = -0.5. \]
Numerical methods

Lax-Friedrich Methods: Finite Difference Method.

Godunov Methods: Finite Volume Method.

ENO and WENO Schemes, and various other higher order methods.

Front Tracking Methods: Use piecewise constant approximation. Treat all fronts (jumps) as shock or small rarefactions. Track all fronts. Fronts may interact or merge. Solve a new Riemann problem.

Challenge:
Accurately approximate shocks as well as the smooth part of the solution.
Front tracking: \[ u_t + \left( u^2(1 - u^2) \right)_x = 0, \quad u(0, x) = \sin(\pi x) \]
References

