Theorem. For the discrete dynamical system
\[
\begin{align*}
    x(n + 1) &= Ax(n), \\
    x(0) &= x_0,
\end{align*}
\]
the vector 0 is asymptotically stable if and only if all eigenvalues \( \lambda_i \) of \( A \) are strictly less than 1 in absolute value:
\[|\lambda_i| < 1.\]

Proof of the "only if" part. Suppose there is an eigenvalue \(|\lambda| \geq 1\) with an eigenvector \( v \). If \( \lambda \) is real choose \( x_0 = v \). Then
\[|x(k)| = |\lambda|^k |v| \not\to 0.\] (1)

If \( \lambda \) is complex and we can take complex initial conditions, then the above argument (1) still holds. If we restrict ourselves to the case when initial conditions must be real then the proof is a bit more involved. The basic idea is still use (1). Suppose \( \lambda = a + bi \) is complex then the corresponding eigenvector \( v = u + wi \) is also complex. Moreover the complex conjugate \( \bar{\lambda} = a - bi \) is an eigenvalue with the eigenvector \( \bar{v} = u - wi \). Denote
\[\lambda = r (\cos \theta + i \sin \theta), \ r \geq 1.\]

Then
\[\lambda^k = r^k (\cos k\theta + i \sin k\theta), \ \bar{\lambda}^k = r^k (\cos k\theta - i \sin k\theta).\]

Consider real and imaginary parts of \( v \):
\[u = 1/2(v + \bar{v}), \ w = 1/(2i)(v - \bar{v}).\]
\[A^k u = 1/2 A^k (v + \bar{v}) = 1/2 \left( \lambda^k v + \bar{\lambda}^k \bar{v} \right) = r^k (u \cos k\theta - v \sin k\theta).\]
\[A^k w = r^k (v \cos k\theta + u \sin k\theta).\]

We complete the proof by contradiction. Assume both \( |A^k w| \) and \( |A^k u| \) simultaneously converge to zero. Then
\[|\lambda^k||v| = |A^k v| = \sqrt{|A^k w|^2 + |A^k u|^2} \to 0,\]
but this contradicts (1).

Continuous dynamical systems

- Comparison of discrete and continuous dynamical systems
- ODEs and matrix exponential
- Stability
- Extra material. Matrix exponential of a Jordan block
Discrete
\[
\begin{align*}
\begin{cases}
x(n+1) = Ax(n), \\
x(0) = x_0.
\end{cases}
\end{align*}
\tag{2}
\]

Continuous
\[
\begin{align*}
\begin{cases}
\frac{dx(t)}{dt} = Ax(n), \\
x(0) = x_0.
\end{cases}
\end{align*}
\tag{3}
\]

Example
\[
\begin{align*}
\begin{cases}
\frac{dx_1(t)}{dt} &= 3x_1 + x_2, \\
\frac{dx_2(t)}{dt} &= x_1 + 3x_2.
\end{cases}
\end{align*}
\tag{4}
\]

Discrete trajectory \((x(0), x(1), \ldots, x(n), \ldots)\).
Continuous trajectory \(x(t), t \in [0, \infty]\).

Space of discrete/continuous trajectories is a Linear Space.

The set of all solutions of \((2)\) is a Linear Subspace of all discrete trajectories.
The set of all solutions of \((3)\) is a Linear Subspace of all continuous trajectories.

If \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda\), then the trajectory of the discrete dynamical system \((2)\) with \(x_0 = v\) is
\[
x(k) = \lambda^k v.
\]

If \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda\), then the trajectory of the discrete dynamical system \((3)\) with \(x_0 = v\) is
\[
x(k) = \exp(\lambda t) v.
\]

For \((4)\) let
\[
x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Then
\[
x(t) = \exp(4t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

**Theorem.** Suppose \(A\) is diagonalizable \(A = VDV^{-1}\) where
\[
D = \begin{pmatrix}
d_1 & 0 & \cdots & 0 \\
0 & d_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & d_n
\end{pmatrix}
\]

Then the solution of \((3)\) is
\[
x(t) = V \exp(Dt)V^{-1}x_0,
\]
where
\[
\exp(Dt) = \begin{pmatrix}
\exp(d_1 t) & 0 & \cdots & 0 \\
0 & \exp(d_2 t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \exp(d_n t)
\end{pmatrix}
\]
Proof: Let
\[ x_0 = \sum c_i y_i \]
where \( y_i \) are eigenvectors with eigenvalues \( d_i \) respectively. And
\[
V : e_i \to y_i, \ e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\]

Then
\[ z(t) = \sum c_i \exp(d_i t) y_i. \]

Or in matrix notation
\[ z(t) = V \exp(Dt)V^{-1}z_0 \]
Why?

Hence we are able to define
\[ \exp(At) = V \exp(Dt)V^{-1}. \]

**Stability**

*Theorem* A continuous dynamical system (3) is stable if and only if all eigenvalues of \( A \) satisfy
\[ |\exp(\lambda_i)| \leq q < 1, \]
or (equivalently) real parts of eigenvalues are negative.

*Proof* is identical to the discrete case (How?)

**Extra Material**

Matrix Exponential of a Jordan block.

Suppose instead of \( D \) we have a Jordan matrix
\[
J = \begin{pmatrix}
J_1 & 0 & \ldots & 0 \\
0 & J_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_m
\end{pmatrix}.
\]

As in the case of difference equations, the question of finding \( \exp(At) \) reduces to question of finding \( \exp(Jt) \), where the \( k \times k \) Jordan block is
\[
J = \begin{pmatrix}
\lambda & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & \lambda & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & \lambda & 1 & \ldots & 0 & 0 \\
0 & 0 & 0 & \lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & \lambda
\end{pmatrix}.
\]
We claim that

\[ \exp(Jt) = \begin{pmatrix}
\exp(\lambda t) & t \exp(\lambda t) & \frac{t^2}{2} \exp(\lambda t) & \frac{t^3}{3!} \exp(\lambda t) & \cdots & \frac{t^{k-1}}{(k-1)!} \exp(\lambda t) & \frac{t^k}{k!} \exp(\lambda t) \\
0 & \exp(\lambda t) & t \exp(\lambda t) & \frac{t^2}{2} \exp(\lambda t) & \cdots & \frac{t^{k-2}}{(k-2)!} \exp(\lambda t) & \frac{t^{k-1}}{(k-1)!} \exp(\lambda t) \\
0 & 0 & \exp(\lambda t) & t \exp(\lambda t) & \cdots & \frac{t^{k-3}}{(k-3)!} \exp(\lambda t) & \frac{t^{k-2}}{(k-2)!} \exp(\lambda t) \\
0 & 0 & 0 & \exp(\lambda t) & \cdots & \frac{t^{k-4}}{(k-4)!} \exp(\lambda t) & \frac{t^{k-3}}{(k-3)!} \exp(\lambda t) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \exp(\lambda t)
\end{pmatrix}, \]

The easiest way to show that is by (directly) solving the equation

\[ x' = Jx \text{ How?} \]

where

\[ x' = \frac{x(t)}{dt}. \]

A more involved, but worthy path is to prove

\[ \exp(Jt) = I + Jt + J^2 \frac{t^2}{2} + J^3 \frac{t^3}{3!} + \ldots \]

by observing that the right-hand side and the left-hand side are the only two (families of) matrices, say, \( A(t) \) such that

\[ A(0) = I, \lim_{\Delta t \to 0} \frac{A(t + \Delta t) - A(t)}{\Delta t} = JA(t). \]