In my previous article [Ba2] I discussed the fundamental and characteristic properties of complete Riemannian manifolds of nonpositive sectional curvature. In the present article, I want to introduce the reader into the theory of metric spaces of nonpositive curvature and, more generally, spaces with upper curvature bounds. I will also discuss a few more recent results.

The foundations of the theory of metric spaces with upper curvature bounds were laid in the work of A. D. Alexandrov [Ale1] and H. Busemann [Bus], some ideas can be traced back to the work of A. Wald [Wa]. More recent treatments are contained in [Gr2], [BriHa], [KlLe], and [Ba2]. The reader is referred to these for more details. An exhaustive treatment of the foundations and some interesting applications are contained in the monograph of Rinow [Ri]. Riemannian manifolds of nonpositive sectional curvature are discussed in [BaGrSr] and in the more recent monograph [Eb3] of Eberlein.

The work of Gromov on geometric group theory, see e.g. [Gr2, Gr3], and of Gromov and Schoen on the rigidity and arithmeticity of lattices [GrSn] have renewed the interest in the theory of spaces with upper curvature bounds and led to new developments and further applications. The main idea is to axiomatize properties of Riemannian manifolds with upper curvature bounds on their sectional curvature, notably properties related to trigonometry.\footnote{There is also a theory of spaces with lower curvature bounds, but it has a completely different flavor. Note that there is an important distinction between lower and upper curvature bounds: Whereas the former are stable under Gromov-Hausdorff limits, the latter are not.} In Riemannian geometry these properties constitute the contents of triangle comparison theory. Many results in global Riemannian geometry rely on triangle comparison, starting from the sphere theorem up to some of the most recent results.

It follows from the setup that arguments and results from Riemannian geometry which, except for some very basic geometry of geodesics, depend only on triangle comparison, should have their counterpart in the theory of spaces with upper curvature bounds. One very important example, where this holds, concerns the theorem of Hadamard and Cartan on simply connected complete Riemannian manifolds of nonpositive sectional curvature, see Theorem 1.8. For this reason, the theory of spaces of nonpositive curvature is of particular importance. The range of applications includes...
geometric group theory, lattices in semisimple Lie groups and the theory of Tits buildings.

This article is based on my talk at the DMV meeting in Mainz in 1999. In the talk, I concentrated on problems related to the so called geometric rank of spaces. In this article I added a few comments and references to other directions which have been followed successfully. However, the article does not give a complete picture of what has been achieved in the field of metric spaces with curvature bounded from above.

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1. The Setup

We say that a metric space is a length space if the distance between any two points \( x, y \in X \) is given by the infimum over the lengths of curves connecting \( x \) with \( y \). If \( X \) is complete and if, for all \( x, y \in X \) and \( \varepsilon > 0 \), there is a point \( z \in X \) with

\[
\max\{d(x, z), d(y, z)\} < \frac{1}{2}d(x, y) + \varepsilon,
\]

then \( X \) is a length space. Length spaces are path connected.

We say that a curve \( c : I \rightarrow X \) is a minimal geodesic if \( c \) is parameterized proportionally to arc length and if \( c|[s, t] \) realizes the distance between \( c(s) \) and \( c(t) \) for all \( s, t \in I \). We say that a curve \( c \) in \( X \) is a geodesic if \( c \) is a minimal geodesic locally. We say that \( X \) is a geodesic space if the distance between any two points of \( X \) is realized by a minimal geodesic.

For a connected Riemannian manifold, completeness and geodesic completeness are equivalent. Of course, this fails in the general context of geodesic spaces. For example, a compact interval is complete, but it is certainly not geodesically complete. The following general version of the Hopf-Rinow Theorem adresses the completeness question:

1.1. Theorem (Cohn-Vossen [CoVo, Ba2]). For a locally compact length space \( X \), the following assertions are equivalent:

1. \( X \) is complete.
2. Any geodesic \( c : [0, 1) \rightarrow X \) can be extended to \([0, 1]\).
3. For some point \( x \in X \), any minimal geodesic \( c : [0, 1) \rightarrow X \) with \( c(0) = x \) can be extended to \([0, 1]\).
4. Bounded subsets of \( X \) are relatively compact.

Each of these implies that \( X \) is geodesic.

Geodesic completeness comes as an extra assumption. Hairy spaces as in Example 2) below and other examples show that geodesic completeness is actually related to the regularity of the underlying space. For example, a finite simplicial complex \( X \) with a
spherical, Euclidean, or hyperbolic structure as in Example 3) below is geodesically complete if and only if it does not contain boundary simplices.

We say that a map \( f : X \to Y \) between metric spaces \( X \) and \( Y \) is a \textit{quasi-isometric embedding} if there are constants \( A \geq 0 \) and \( L \geq 1 \) such that

\[
(1.2) \quad L^{-1}d(x, x') - A \leq d(f(x), f(x')) \leq Ld(x, x') + A
\]

for all \( x, x' \in X \). We say that a quasi-isometric embedding is a \textit{quasi-isometry} if the image of \( f \) is \( R \)-dense in \( Y \) for some constant \( R \geq 0 \). Quasi-isometries play an important role in Mostow’s proof of his celebrated rigidity theorem.

Note that quasi-isometries are not required to be continuous. In fact, one of the important ideas in the context of quasi-isometries is contained in the following elementary result, in which continuity would be out of place.

1.3. THEOREM (Gromov [Gr1]). Suppose a group \( \Gamma \) acts properly discontinuously and cocompactly on a complete length space \( X \). Then \( \Gamma \) is finitely presented and, when endowed with a word metric, is quasi-isometric to \( X \).

For this and other reasons, quasi-isometry invariants are important in geometry and group theory. An example of such an invariant is hyperbolicity in the sense of Gromov. We will see others further on.

\textbf{Trigonometry and Curvature.} It will be convenient to relax the notion of geodesic spaces somewhat. For \( D > 0 \) given, we say that \( X \) is \( D \)-\textit{geodesic} if the distance between any two points of \( X \) of distance \(< D \) is realized by a minimal geodesic.

For \( \kappa \in \mathbb{R} \), let \( M^n_\kappa \) be the \textit{model space} of dimension \( n \) and constant sectional curvature \( \kappa \): \( M^n_\kappa \) is the sphere of radius \( 1/\sqrt{|\kappa|} \) in Euclidean space of dimension \( n + 1 \) if \( \kappa > 0 \), Euclidean space of dimension \( n \) if \( \kappa = 0 \) and \( n \)-dimensional hyperbolic space of curvature \( \kappa \) if \( \kappa < 0 \). We denote by \( D(\kappa) \) the diameter of \( M^n_\kappa \), that is, \( D(\kappa) = \pi/\sqrt{|\kappa|} \) if \( \kappa > 0 \) and \( D(\kappa) = \infty \) if \( \kappa \leq 0 \).

A \textit{geodesic triangle} in \( X \) consists of geodesic segments \( c_1, c_2, c_3 : [0, 1] \to X \) whose endpoints match in the usual way. A geodesic triangle \( \Delta = (c_1, c_2, c_3) \) is \textit{admissible} if

\[
(1.4) \quad L(c_i) \leq L(c_j) + L(c_k)
\]

for all pairwise different \( i, j, k \in \{1, 2, 3\} \). A geodesic triangle \( \Delta = (c_1, c_2, c_3) \) is admissible if its sides \( c_1, c_2, \) and \( c_3 \) are minimal.

Let \( \Delta = (c_1, c_2, c_3) \) be a geodesic triangle in \( X \). Then a \textit{comparison triangle} for \( \Delta \) in \( M^n_\kappa \) is a geodesic triangle \( \bar{\Delta} = (\bar{c}_1, \bar{c}_2, \bar{c}_3) \) in \( M^2_\kappa \) with lengths \( L(\bar{c}_i) = L(c_i) \) for \( i = 1, 2, 3 \). A comparison triangle exists and is unique up to congruence if \( \Delta \) is admissible and the \textit{circumference}

\[
(1.5) \quad L(c_1) + L(c_2) + L(c_3) < 2D(\kappa).
\]

\textsuperscript{2}A simplex of \( X \) of dimension \( k \) is a boundary simplex of \( X \) if it is adjacent to precisely one simplex of \( X \) of dimension \( k + 1 \).

\textsuperscript{3}In the early sixties, F.John used the term \textit{quasi-isometry} to denote a smooth map whose differential is uniformly bounded from above and below by some positive constants. This was pointed out to me by S.Hildebrandt, compare [Hi] for a discussion. The definition here seems to go back to Mostow [Mo].
Let \( \kappa \in \mathbb{R} \). We say that \( X \) is a CAT(\( \kappa \))-space if \( X \) is D(\( \kappa \))-geodesic and if, for any admissible triangle \( \Delta = (c_1, c_2, c_3) \) in \( X \) of circumference \( < 2D(\kappa) \) and comparison triangle \( \bar{\Delta} = (\bar{c}_1, \bar{c}_2, \bar{c}_3) \) in \( M^2_\kappa \), we have
\[
(1.6) \quad d(c_i(s), c_j(t)) \leq d(\bar{c}_i(s), \bar{c}_j(t))
\]
for all \( i, j \in \{1, 2, 3\} \) and \( s, t \in [0, 1] \). It is only necessary to require (1.6) for a vertex and the midpoint of the opposite side, that is, for \( s = 0 \) and \( t = 1/2 \). In a CAT(\( \kappa \))-space, geodesics of length less than D(\( \kappa \)) are minimal and the unique geodesic connection of their endpoints. In particular, all geodesic triangles in a CAT(\( \kappa \))-space of circumference less than 2D(\( \kappa \)) have minimal sides and hence are admissible.

A complete metric space is CAT(0) if and only if for each pair of points \( x, y \in X \) there is a point \( m \in X \) such that
\[
(1.7) \quad 2d^2(m, z) \leq d^2(x, z) + d^2(y, z) - \frac{1}{2}d^2(x, y)
\]
for all points \( z \in X \). This is the property established and used by Bruhat and Tits in their work on Euclidean buildings [BruTi], see also [Bro]. Note that (1.7) corresponds to (1.6), where \( \kappa = 0 \), \( s = 0 \), and \( t = 1/2 \). The point is that now it is not assumed a priori that \( X \) is geodesic, this is a consequence.

Suppose that \( X \) is a CAT(\( \kappa \))-space and let \( \Delta = (c_1, c_2, c_3) \) be a geodesic triangle in \( X \) of circumference \( < 2D(\kappa) \). Then if there is equality in (1.6) for some pair \( c_i(s), c_j(t) \), where \( c_j(t) \) is not contained in the image of \( c_i \), then the convex hull of \( \Delta \) in \( X \) is isometric to the convex hull of the comparison triangle \( \bar{\Delta} \) in \( M^2_\kappa \). This rigidity of geodesic triangles is at the base of most of the rigidity results of spaces with upper curvature bounds.

Curvature is a local phenomenon
\(^4\): Let \( X \) be a metric space and \( x \in X \) be a point. We define the upper curvature \( K^+_X(x) \) of \( X \) at \( x \) to be the infimum over all \( \kappa \) such that \( x \) has a neighborhood which is CAT(\( \kappa \)). We say that \( X \) has curvature bounded from above by \( \kappa \) if \( K^+_X \leq \kappa \)
\(^5\). It is left as a little exercise that this definition is equivalent to the usual one.

A D(\( \kappa \))-geodesic space \( X \) of curvature bounded from above by \( \kappa \) is a CAT(\( \kappa \))-space if and only if it has injectivity radius \(^6\) D(\( \kappa \)).

**Examples.** 1) The definition is modelled after Riemannian manifolds with upper curvature bounds: A standard triangle comparison theorem says that a Riemannian manifold has sectional curvature bounded from above by \( \kappa \) if and only if \( K^+_X(x) \leq \kappa \).

\(^4\)This point of view is opposite to the point of view maintained in some publications, where curvature is defined to be a global phenomenon.

\(^5\)The corresponding definition of lower curvature bounds is essentially obtained by reversing (1.6). An important difference between upper and lower curvature bounds is reflected by the differences in the assumptions of the corresponding triangle comparison theorems in Riemannian geometry.

\(^6\)The injectivity radius of a space \( X \) is defined to be the supremum over all \( r \geq 0 \) such that for each pair of points in \( X \) of distance \( < r \), there is a unique minimizing geodesic connecting them.
SPACES OF NONPOSITIVE CURVATURE

Riemannian symmetric spaces of noncompact type and their quotients are the most important examples of Riemannian manifolds of nonpositive curvature.

2) Suppose \( X \) is a graph, say locally finite. Assume that each edge \( \sigma \) of \( X \) is endowed with a metric \( d_\sigma \) such that \( (\sigma, d_\sigma) \) is isometric to some compact interval. Then \( X \) with the induced length metric \( d \) has curvature bounded from above by \( \kappa \) for arbitrary \( \kappa \). In this sense, the curvature of \( X \) is \(-\infty\).

If \( X \) is a space of curvature bounded from above by a constant \( \kappa \) and \( x \in X \) is a chosen point, then we obtain a new complete space \( Y \) of curvature bounded from above by \( \kappa \) by attaching a compact interval \( I \) to \( X \) by identifying one of its end points with \( x \). We obtain hairy examples.

3) Suppose \( X \) is a simplicial complex; for the sake of simplicity assume \( X \) to be locally finite. Suppose each simplex \( \sigma \) of \( X \) is endowed with a metric \( d_\sigma \) such that \( (\sigma, d_\sigma) \) is isometric to a standard simplex of curvature \( \kappa \), that is, \( \sigma \) is isometric to an intersection \( \bar{\sigma} \) of \( k+1 \) halfspaces in \( M^k_{\kappa} \) in general position, where \( k = \dim \sigma \). Suppose furthermore that \( d_\sigma|_\eta = d_\eta \) whenever \( \eta \) is adjacent to \( \sigma \). Such a structure on \( X \) will be called a piecewise spherical, Euclidean, or hyperbolic structure on \( X \) for \( \kappa = 1, 0, \) or \(-1\), respectively.

Let \( v \) be a vertex of \( X \). Then we may view any simplex \( \eta \) of the link \( L_v \) of \( v \) as the piece of the unit sphere inside the tangent space \( T_v \sigma \) cut out by \( \sigma \), where \( \sigma \) is the simplex of \( X \) determining \( \eta \). This endows \( L_v \) with a piecewise spherical metric.

Suppose \( X \) is endowed with a piecewise spherical, Euclidean, or hyperbolic structure. Then \( X \) with the induced length metric \( d \) has curvature bounded from above by \( 1, 0, \) or \(-1\), respectively, if and only if, for each vertex \( v \) of \( X \), the link \( L_v \), endowed with the induced piecewise spherical metrics on its simplices, is \( \text{CAT}(1) \).

The natural piecewise spherical, Euclidean, or hyperbolic structure on a spherical, Euclidean, or hyperbolic building turns it into a \( \text{CAT}(\kappa) \)-space, with \( \kappa = 1, 0, \) or \(-1\), respectively.

4) There is a precise criterion for piecewise smooth metrics on 2-dimensional simplicial complexes to have an upper curvature bound \( \kappa \), see the appendix of [BaBuy]. However, in higher dimensions it is rather unclear whether there are any general criteria besides some obvious sufficient ones.

5) A Banach space has an upper curvature bound (or lower curvature bound, respectively) if and only if it is a Hilbert space. Banach spaces have nonpositive curvature in a weaker sense though; they are convex in the terminology of [AlBi].

**Hadamard Spaces and their Ideal Boundary.** Following the terminology in the smooth case, a **Hadamard space** is a complete \( \text{CAT}(0) \)-space.

The Hadamard-Cartan theorem in Riemannian geometry establishes a local-to-global phenomenon for simply connected and complete Riemannian manifolds of nonpositive sectional curvature. There is the following version in the general setting (compare also [AlBi], where Alexander and Bishop obtain a version for convex geodesic spaces).

1.8. **Theorem.** A simply connected and complete space of nonpositive curvature is a Hadamard space.
Any geodesic segment in a Hadamard space is minimal and is, in fact, the unique geodesic connection of its endpoints. In particular, Hadamard spaces are contractible: Fix an origin \( x \in X \). For \( y \in X \) let \( c_{xy} : [0,1] \to X \) be the unique geodesic with \( c(0) = x \) and \( c(1) = y \). Then \( C(t,y) := c_{xy}(1-t), 0 \leq t \leq 1 \), defines a contraction of \( X \) to the origin \( x \). It follows that complete spaces of nonpositive curvature are \( K(\pi, 1) \)-spaces.

In a Hadamard space, all reasonable functions are convex. For example, the distance to a convex subset is a convex function. For any two geodesics \( c_0, c_1 \), the function \( d(c_0(s), c_1(t)) \) is convex on its domain of definition in the \((s,t)\)-plane.

There is an important compactification of locally compact Hadamard spaces by ideal points, corresponding to the points on the unit circle in the case of the Poincaré disc model of the hyperbolic plane: Let \( X \) be a Hadamard space. Say that unit speed geodesic rays \( c_0, c_1 : [0, \infty) \to X \) are asymptotic if the distance \( d(c_0(t), c_1(t)) \) is uniformly bounded. The following are elementary:

(AR1) For any unit speed geodesic ray \( c : [0, \infty) \to X \) and any point \( x \in X \), there is a unique unit speed geodesic ray \( c_x : [0, \infty) \to X \) asymptotic to \( c \) with \( c_x(0) = x \).

(AR2) For any sequence \( c_n : [0, \infty) \to X \) of unit speed geodesic rays in \( X \) converging to a unit speed geodesic ray \( c : [0, \infty) \to X \) and any sequence \( x_n \) of points in \( X \) converging to a point \( x \in X \), the sequence of unit speed geodesic rays \( c_n x_n : [0, \infty) \to X \) converges to \( c_x : [0, \infty) \to X \).

We let \( \partial X \) be the set of asymptote classes of unit speed geodesic rays in \( X \) and set \( \bar{X} := X \cup \partial X \). The cone topology on \( \bar{X} \) has the following two characteristic properties:

(CT1) It coincides with the given topology on \( X \).

(CT2) A sequence \( \xi_n \) in \( \bar{X} \) converges to a point \( \xi \in \partial X \) if and only if, for any \( x \in X \), the unit speed geodesic from \( x \) to \( \xi_n \) (the ray from \( x \) in \( \xi_n \) if \( \xi_n \in \partial X \)) converges to the unit speed geodesic ray from \( x \) representing \( \xi \).

In the case of Hadamard manifolds, the cone topology turns \( \partial X \) into a sphere, the sphere at infinity, and \( \bar{X} \) is homeomorphic to a closed ball. In the general setting, the topology is more complicated. However, as long as \( X \) is locally compact, \( \bar{X} \) is a compactification of \( X \).

There is a second topology on \( \partial X \): For \( \xi, \eta \in \partial X \) and \( x \in X \), let \( c_\xi \) and \( c_\eta \) be the unit speed geodesic rays starting at \( x \) and representing \( \xi \) and \( \eta \), respectively. Let \( d = \lim d(c_\xi(t), c_\eta(t))/t \) and \( \Delta \) be the Euclidean triangle with two sides of length 1 and one side of length \( d \). Let \( \alpha = \alpha(\xi, \eta) \) be the angle at the vertex of \( \Delta \) opposite the side of length \( d \). Then \( \alpha \) is a metric on \( \partial X \). The associated length metric is called the Tits metric. The Tits metric is nontrivial if there is enough flatness in \( X \).

It is more or less clear that the action of the group of isometries of \( X \) extends continuously to \( \bar{X} \) in the cone topology and that the restriction of this action to \( \partial X \) is also continuous in the Tits metric.
It is too optimistic to expect that $X$ is determined by the structure of $\partial X$. An exception is Leeb’s characterization of symmetric spaces and Euclidean buildings, Theorem 3.2 below. In general, there is the following natural

1.9. QUESTION. How much of the geometry of a Hadamard space $X$ is caught by the structure of $\partial X$?

There is interesting work of, among others, Buyalo [Buy], Croke and Kleiner [CrKl1, CrKl2] and Hummel and Schroeder [HuSr], where this question is discussed.

2. LARGE ISOMETRY GROUPS AND GEOMETRY

In Riemannian geometry there are many results relating the geometry and topology of manifolds of nonpositive sectional curvature. There are particularly close relations in the compact case. As a rule, the results whose proof does not involve the Margulis Lemma also hold for compact spaces of nonpositive curvature. Some care has to be taken though since such spaces might be infinite dimensional or might be irregular in one way or another. Another important feature, difficult to handle and not present in the smooth case, is the branching of geodesics. For trees and simplicial spaces with piecewise smooth metrics, branching of geodesics occurs along simplices of positive codimension.

We will formulate our results in terms of the universal covering space, a Hadamard space in the induced metric structure, and the group of covering transformations, which acts by isometries. The situation we discuss is a bit more general actually: Unless otherwise specified we will assume, in this section, that $X$ is a locally compact Hadamard space and $\Gamma$ a properly discontinuous and cocompact group of isometries.

Groups and Flats. A subset $F \subset X$ is a $k$-flat if $F$ is closed, convex, and isometric to Euclidean space of dimension $k$. A 0-flat is a point, a 1-flat is a complete geodesic. The existence of one or many flats of dimension $k \geq 2$ is a central issue in rigidity questions. The following result is due to Eberlein in the smooth case [Eb1], a similar argument works in the general setting, see [Gr2, Bri1].

2.1. THEOREM. Let $X$ and $\Gamma$ be as above. Then $X$ and $\Gamma$ are hyperbolic (in the sense of Gromov) if and only if $X$ does not contain a flat of dimension 2.

Hyperbolicity is a quasi-isometry invariant. Thus the existence of a 2-flat in $X$ is a quasi-isometry invariant of $X$. In fact, it was shown by Anderson and Schroeder that the existence of a $k$-flat in a compact Riemannian manifold of nonpositive sectional curvature is a quasi-isometry invariant [AnSr]. A new argument by Kleiner allowed the extension to the general setting.

2.2. THEOREM (Kleiner [Kl]). Let $X$ and $\Gamma$ be as above. Then $X$ contains a $k$-flat if and only if there is a quasi-isometric embedding of $k$-dimensional Euclidean space into $X$. 

The flat torus theorem says that for a subgroup $\Delta \subset \Gamma$ isomorphic to $\mathbb{Z}^k$, there is a $\Delta$-invariant $k$-flat $F$ in $X$ such that $\Delta$ acts on $F$ as a cocompact lattice of translations. Hence $X$ is not hyperbolic if $\Gamma$ contains a subgroup isomorphic to $\mathbb{Z}^2$. Thus the previous two results lead to the following

2.3. QUESTION. Is hyperbolicity equivalent to the non-existence of a subgroup of $\Gamma$ isomorphic to $\mathbb{Z}^2$? More generally, does $\Gamma$ contain a subgroup isomorphic to $\mathbb{Z}^k$ if $X$ contains a $k$-flat?

By the work of Bangert and Schroeder [BanSr] the answer is positive in the case of compact, real analytic Riemannian manifolds. Except for this, the answers to these questions are completely open, even in the case where $X$ is a geodesically complete and piecewise Euclidean complex of dimension two!

Related to the flat torus theorem and generalizing earlier results of Avez and Zimmer, there is a fixed point theorem of Burger and Schroeder, see [BugSr]. The extension of their result to the general setting reads as expected; however, the arguments require a nontrivial change, compare [AdBa]:

2.4. THEOREM. Let $X$ be a locally compact Hadamard space and $\Delta$ be an amenable group of isometries of $X$. Then $X$ contains a $\Delta$-invariant flat or $\Delta$ fixes a point of $\bar{X}$.

The two possibilities in the theorem are not mutually exclusive. Maybe there is a more precise version of Theorem 2.4 if $\Delta$ is assumed to be discrete.

**Geometric Rank and Rank Rigidity.** We say that a Hadamard space $X$ has geometric rank $k$ if each geodesic segment of $X$ is contained in a $k$-flat. Note that $X$ is geodesically complete if and only if $X$ has geometric rank $\geq 1$. The geometric rank of a symmetric space of noncompact type coincides with its usual rank. For example, $\text{SL}(n, \mathbb{R})/\text{SO}(n)$ has rank $n - 1$. The geometric rank of a Euclidean building is equal to its dimension. The geometric rank of a product is equal to the sum of the geometric ranks of the factors.

The following result was proved by Eberlein in the smooth case [Eb2], the easy extension of his arguments to the general setting is contained in [BaBr2].

2.5. THEOREM. For $X$ and $\Gamma$ as above, to be of geometric rank $\geq 2$ is a quasi-isometry invariant of $X$ (and $\Gamma$).

Suppose for the moment that $X$ is a smooth Riemannian manifold. Then there are two cases: Either the geometric rank of $X$ is one and then $X$ has significant similarities with manifolds of strictly negative sectional curvature. Or the geometric rank of $X$ is at least two. Then $X$ is a Riemannian product or a symmetric space of noncompact type by the rank rigidity theorem.

For a compact proof of the rank rigidity theorem in the smooth case, see [Ba2].

Here is a list (not quite complete) of properties of rank one manifolds.

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7In the homogeneous case, which has almost empty intersection with the case discussed here, rank rigidity is proved in [He].
2.6. **Theorem.** Suppose $M = X/\Gamma$ is a compact Riemannian manifold of nonpositive sectional curvature. If the geometric rank of $X$ is one, then the following hold:

1. The geodesic flow of $M$ is topologically transitive.
2. Hyperbolic closed geodesics are dense in the tangent bundle of $M$.
3. If $N_h(L)$ denotes the number of geometrically distinct hyperbolic closed geodesics of length at most $L$, then
   $$\lim_{L \to \infty} \frac{1}{L} \ln N_h(L) = h,$$
   where $h > 0$ is the topological entropy of the geodesic flow of $M$ (Knieper [Kn1], for a more precise estimate, see [Kn2]).
4. The Dirichlet problem at infinity is solvable in $X$ (see [Ba2]).
5. The Poisson boundary of $X$ is canonically isomorphic to $X(\infty)$ (see [BaLd]).

None of these properties holds if the geometric rank of $X$ is $\geq 2$.

There should be similar results in the general case. So far, there is only a complete picture for compact 2-dimensional polyhedra with piecewise smooth metrics of nonpositive curvature.

2.7. **Theorem** ([BaBr2]). Let $X$ be a 2-dimensional polyhedron without boundary with a piecewise smooth metric of nonpositive curvature and $\Gamma$ be a properly discontinuous and cocompact group of isometries of $X$. Then we have:

If the geometric rank of $X$ is one, then

1. the geodesic flow of $X$ is topologically transitive mod $\Gamma$.
2. hyperbolic $\Gamma$-closed geodesics are dense in the space of geodesics and their growth rate is equal to the topological entropy of the geodesic flow.

If the geometric rank of $X$ is two, then

1. $X$ is a product of trees or
2. $X$ is a thick Euclidean building.

The easy part of the proof consists in showing that $X$ is a product of trees if the geometric rank of $X$ is two. The hard part of the proof concerns the existence of a hyperbolic $\Gamma$-closed geodesic in the case when the geometric rank is one — a closing lemma in the case of a nonhyperbolic flow.

2.8. **Remark.** For applications in geometric group theory and topology it is important to note that in the case where $X$ has boundary, there is a $\Gamma$-equivariant homotopy equivalence to a polyhedron $Y$ of dimension $\leq 2$ without boundary, on which the action of $\Gamma$ is also properly discontinuous and cocompact and which admits a piecewise smooth $\Gamma$-invariant metric of nonpositive curvature, see [BaBr2].

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8The ergodicity of the geodesic flow is still an open problem. The identical gap in the proofs in [Pe], [Bur], and [BaBr1] has not been filled in yet.

9By definition, the geodesic flow $g^t$ acts tautologically on the space of complete (unit speed) geodesics, $g^t(c)(s) := c(s + t)$. 
The situation in higher dimensions is unclear. Besides the work of Leeb and of Charney and Lytchak, which addresses geometric characterizations of buildings, see Section 3, there is the following partial result in dimension three.

2.9. **Theorem** ([BaBr3, BaBr4]). Let \(X\) be a 3-dimensional polyhedron without boundary, endowed with a piecewise Euclidean structure of nonpositive curvature. Suppose that \(X\) admits a properly discontinuous and cocompact group \(\Gamma\) of isometries. Then if \(X\) has geometric rank \(\geq 2\), then \(X\) is a product or a thick Euclidean building.

The arguments are somewhat complicated, if not unattractive, and seem to need the assumption on the dimension. Moreover, the theorem only addresses the characterization of buildings. What is completely missing is the rank one side of the picture. The most important question here is whether \(X\) admits a (one!) hyperbolic \(\Gamma\)-closed geodesic if the geometric rank of \(X\) is one. The existence of such a geodesic has been established in the case where some link of \(X\) is not connected or has radius \(> \pi\), see [BaBr2, Lemma 7.2].

A closing argument shows that the existence of a hyperbolic \(\Gamma\)-closed geodesic follows from the existence of a hyperbolic \(\Gamma\)-recurrent geodesic, see [Ba2, Section III.3]. The latter would follow if the geodesic flow of \(X\) had an invariant measure which is finite mod \(\Gamma\) and positive on open sets of geodesics. The Liouville measure, as considered in [BaBr2], does not have this property (in general), “it does not see” the most hyperbolic part of the geodesic flow, created by links of diameter \(> \pi\).

3. Other Topics

In my talk at the DMV meeting in Mainz, I did not have the time to survey and explain all the ongoing activities in the field of nonpositively curved spaces. I take this opportunity to mention a few other directions and results which I find particularly fascinating and interesting. I apologize for omissions.

**Rigidity of Symmetric Spaces and Buildings without Groups.** Pansu showed that a quasi-isometry of a quaternionic hyperbolic space or the Cayley hyperbolic plane lies within bounded distance to an isometry [Pa]. Margulis conjectured that the same holds for irreducible symmetric spaces of noncompact type and rank at least two. Kleiner and Leeb settled Margulis’ conjecture and, in fact, proved the following wonderful result.

3.1. **Theorem** (Kleiner-Leeb [KLe]). Let \(X\) and \(Y\) be irreducible symmetric spaces of noncompact type of rank at least 2 or irreducible thick Euclidean buildings of rank at least 2 with cocompact affine Weyl group and Moufang Tits boundary. Then any quasi-isometry \(f : X \to Y\) is within bounded distance to a homothety.

\[^{10}\text{It is actually sufficient to show the existence of a } \Gamma\text{-closed geodesic which does not bound a flat half plane. Compare the discussion in [Ba2, Section III.3]. I proved this in the case where all links of } \X\text{ have diameter } \pi, \text{ a rather restrictive assumption.}\]
In Mostow’s proof of his celebrated rigidity theorem, quasi-isometries occur as lifts of homotopy equivalences of compact quotients of $X$ and $Y$. In particular, they are equivariant with respect to large groups of isometries of $X$ and $Y$. In contrast to this, the theorem of Kleiner and Leeb does not assume any equivariance of the quasi-isometry.

There is also the following beautiful geometric characterization of symmetric spaces and Euclidean buildings.

3.2. **Theorem** (Leeb [Le]). Let $X$ be a geodesically complete and locally compact Hadamard space. Assume that $\partial \mathbb{T}X$ is an irreducible thick spherical building. Then $X$ is a Riemannian symmetric space or a Euclidean building.

This result also allows to remove the assumption that the Tits boundary be Moufang from Theorem 3.1.

Another very nice and useful geometric characterization of buildings is due to Charney and Lytchak (there is earlier unpublished work of Kleiner). Let $X$ be a geodesically complete geodesic space. Say that $X$ has the discrete extension property if, for each geodesic segment, the set of its geodesic extensions is discrete.

3.3. **Theorem** (Charney-Lytchak [ChLy]). Let $X$ be a connected simplicial complex of dimension $\geq 2$ with a piecewise spherical or Euclidean structure, respectively, and suppose $X$ is geodesically complete and geodesic. Then $X$ is a spherical or metric Euclidean building, respectively, if and only if

1. $X$ has the discrete extension property and
2. $X$ is CAT(1) or CAT(0), respectively.

Here we count trees and Euclidean cones over spherical buildings also as metric Euclidean buildings.

**Geometric Group Theory and Topology.** From the very beginning in the work of Dehn, a considerable part of geometric group theory relies on the geometry of negatively and nonpositively curved spaces. The same seems to be true for geometric topology. For results in these directions, see for example [Bri2, Da, DaMo, Fa, FaJo, GHV, Gr2, Gr3].

**Harmonic Maps and Rigidity.** The concept of a harmonic map into a metric space was developed by Gromov and Schoen [GrSn]. They used it to show that lattices of quaternionic hyperbolic spaces and the Cayley hyperbolic plane are arithmetic. Their theory was taken up and developed further by many mathematicians: Korevaar and Schoen, Jost, and others. For a more recent article with some relevant references, see [Ma].

**Conjugacy of Geodesic Flows.** There are quite a few interesting results about conjugacy and conjugacy rigidity of geodesic flows, see for example [Cr1, Cr2, CrKIEb, Ot]. This is a topic close to the theory of negatively curved spaces — a different world, see for example [Ha] and [BeCoGa].
Negative Curvature. This subfield is certainly larger than the field of nonpositive curvature. The same is true, in a next step, for the field of constant negative curvature.
REFERENCES


MATHEMATHISCHES INSTITUT, UNIVERSITÄT BONN, BERINGSTRASSE 1, D-53115 BONN
E-mail address: ballmann@math.uni-bonn.de